

Optimality conditions for multi-objective fractional semi-infinite programming with uniform $K - (F_b, \rho)$ convexity

Hong Yang, Yongchun He

*School of Mathematics and Statistics,
Yulin University, Yulin 719000, Shanxi, China*

Abstract

Base on some nonsmooth generalized convex functions called uniform $K - (F_b, \rho)$ -convex function, uniform $K - (F_b, \rho)$ -pseudoconvex function, uniform $K - (F_b, \rho)$ -quasiconvex function, nonsmooth multi-objective fractional semi-infinite programming involving these generalized convex functions is researched, and some sufficient optimality conditions are obtained.

Key words: NONSMOOTH, MULTI-OBJECTIVE FRACTIONAL SEMI-INFINITE PROGRAMMING, OPTIMALITY CONDITIONS, UNIFORM $K - (F_b, \rho)$ -CONVEX FUNCTION

Introduction

The convexity theory plays an important role in many aspects in mathematical programming. In recent years, to relax convexity assumption involved in sufficient conditions for optimality or duality theorems, various generalizations of convex functions have appeared in the literature. Hanson and Mond introduced type I and type II function[1]. Reuda and Hanson extended type I function and obtained pseudo type I and quasi type I function[2]. Bector and Singh introduced b -convex function[3]. Bector, Suneja and Gupta extended b -convex function and defined univex function[4]. Mishra discussed the optimality and duality for multi-objective programming with generalized univexity[5]. Preda introduced (F, ρ) -convex function as extension of F -convex function and ρ -convex function[6-8]. Hong Yang defined $K - (F_b, \rho)$ -convex function

and discussed the optimality and duality for multi-objective semi-infinite programming involving these generalized convexity[9-10].

As a branch of optimization, fractional programming has important practical significance in problems such as resource allocation, investment portfolio, etc. So research about fractional programming has attracted a wide spread attention. For example, Bector discussed the optimality and duality for subdifferentiable multi-objective fractional programming [11]. Liu presented three dual model of minimax fractional programming[12]. Kuk researched the optimality and duality for nonsmooth multi-objective fractional programming with generalized invexity[13]. Mishra discussed the duality for nondifferentiable minimax fractional programming involving generalized α -uniform convexity[14]. Ho researched the optimality and duality for

nonsmooth minimax fractional programming involving exponential (p, r) -invexity [15]. Tripathy discussed the mixed type duality for multi-objective fractional programming with generalized ρ -invexity [16].

In this paper, based on a new class of generalized convex functions, that is, uniform $K-(F_b, \rho)$ -convex function, uniform $K-(F_b, \rho)$ -pseudoconvex function, uniform $K-(F_b, \rho)$ -quasiconvex function, we consider nonsmooth multiobjective fractional semi-infinite programming involving these generalized convex functions and obtain some sufficient optimality conditions.

Definitions

Throughout this paper, let R^n be the n dimensional Euclidean space and R_+^n be its nonnegative orthant. Now we consider the following multi-objective fractional semi-infinite programming problem:

$$(VFP) \begin{cases} \min \frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_p(x)}{g_p(x)} \right) \\ \text{s.t. } h(x, u) \leq 0, x \in X, u \in U. \end{cases}$$

where $X \neq \emptyset$ is an open subset of R^n , $f(x) = (f_1(x), f_2(x), \dots, f_p(x)) : X \rightarrow R^p$, $g(x) = (g_1(x), g_2(x), \dots, g_p(x)) : X \rightarrow R^p$, $h : X \times U \rightarrow R$, $U \subset R$ is an infinite parameter set. $\forall x \in X, g(x) > 0$.

Let $X^0 = \{x \mid h(x, u) \leq 0, x \in X, u \in U\}$, $\Delta = \{i \mid h(x, u^i) \leq 0, x \in X, u^i \in U\}$, $I(\bar{x}) = \{i \mid h(\bar{x}, u^i) \leq 0, \bar{x} \in X, u^i \in U\}$, $U^* = \{u^i \in U \mid h(x, u^i) \leq 0, x \in X, i \in \Delta\}$ is any countable subset of U , $\Lambda = \{\mu_j \mid \mu_j \geq 0, j \in \Delta\}$, there is only finite μ_j such that $\mu_j \neq 0$.

Notations. If $x, y \in R^n$, then $x < y \Leftrightarrow x_i < y_i, i = 1, 2, \dots, n$; $x \leq y \Leftrightarrow x_i \leq y_i, i = 1, 2, \dots, n$, and there exists at least one $i_0 \in \{1, 2, \dots, n\}$ such that $x_{i_0} < y_{i_0}$.

Definition 1^[17] Let $K(\cdot, \cdot)$ is a local cone approximation, the function $f^K(x, \cdot) : X \rightarrow R$ with $f^K(x; y) := \inf\{\xi \in R \mid (y, \xi) \in K(\text{epif}, (x, f(x)), y \in R^n)\}$ is called K -directional derivative of f at x .

Definition 2^[17] A function $f : X \rightarrow R$ is called K -subdifferentiable at x if there exists a convex compact set $\partial^K f(x)$ such that

$$f^K(x, y) = \max_{\xi \in \partial^K f(x)} \langle \xi, y \rangle, \forall y \in R^n,$$

where

$$\partial^K f(x) := \{x^* \in X^* \mid \langle y, x^* \rangle \leq f^K(x; y), \forall y \in R^n\}$$

is called K -subdifferential of f at x .

Definition 3 A functional

$F : X \times X \times R^n \rightarrow R (X \subset R^n)$ is called sublinear with respect to the third variable, if for any $x_1, x_2 \in X$,

- (i) $F(x_1, x_2; a_1 + a_2) \leq F(x_1, x_2; a_1) + F(x_1, x_2; a_2), \forall a_1, a_2 \in R^m$;
- (ii) $F(x_1, x_2; ra) = rF(x_1, x_2; a), \forall r \in R, r \geq 0, a \in R^m$.

Definition 4 $x^* \in X^0$ is called an efficient solution for (VFP) if and only if there exists no $x \in X^0$ such that

$$\frac{f(x^*)}{g(x^*)} \leq \frac{f(x)}{g(x)}.$$

Definition 5 $x^* \in X^0$ is called a weak efficient solution for (VFP) if and only if there exists no $x \in X^0$ such that

$$\frac{f(x^*)}{g(x^*)} < \frac{f(x)}{g(x)}.$$

In the following definitions, we suppose $C \subset R^n$ is a nonempty set, $x_0 \in C$, $f : C \rightarrow R$ is a local Lipschitz function at x_0 , $F : C \times C \times R^n \rightarrow R$ is sublinear with respect to the third variable, $\phi : R \rightarrow R, b : C \times C \times [0, 1] \rightarrow R_+$, $\lim_{\lambda \rightarrow 0^+} b(x, x_0; \lambda) = b(x, x_0)$, $d(\cdot, \cdot)$

is a pseudo-metric in R^n . In [17], Elster and Thier-Felder defined K -directional derivative and K -subdifferential and pointed out that K -subdifferential is most generalized. In [9], using K -directional derivative and K -subdifferential, some new generalized convex functions are defined as follow:

Definition 6 A function $f : C \rightarrow R$ is said to be uniform $K-(F_b, \rho)$ -convex at x_0 with respect to F, ϕ, b, d , if for all $x \in C$, there exists $\rho \in R$ such that

$$b(x, x_0)\phi[f(x) - f(x_0)] \geq F(x, x_0; \xi) + \rho d^2(x, y), \forall \xi \in \partial^K f(x_0).$$

Definition 7 A function $f : C \rightarrow R$ is said to be strictly uniform $K-(F_b, \rho)$ -convex at x_0 with respect to F, ϕ, b, d , if for all $x \in C, x \neq x_0$, there exists $\rho \in R$ such that

$$b(x, x_0)\phi[f(x) - f(x_0)] > F(x, x_0; \xi) + \rho d^2(x, x_0), \forall \xi \in \partial^K f(x_0).$$

Definition 8 A function $f : C \rightarrow R$ is said to be uniform $K-(F_b, \rho)$ -pseudoconvex at x_0 with respect to F, ϕ, b, d , if for all $x \in C$, there exists $\rho \in R$ such that

$$b(x, x_0)\phi[f(x) - f(x_0)] < 0 \Rightarrow F(x, x_0; \xi) + \rho d^2(x, y) < 0, \forall \xi \in \partial^K f(x_0).$$

Definition 9 A function $f : C \rightarrow R$ is said to be strictly uniform $K - (F_b, \rho)$ -pseudoconvex at x_0 with respect to F, ϕ, b, d , if for all $x \in C, x \neq x_0$, there exists $\rho \in R$ such that

$$b(x, x_0)\phi[f(x) - f(x_0)] \leq 0 \Rightarrow$$

$$F(x, x_0; \xi) + \rho d^2(x, y) < 0, \forall \xi \in \partial^K f(x_0).$$

Definition 10 A function $f : C \rightarrow R$ is said to be uniform $K - (F_b, \rho)$ -quasiconvex at x_0 with respect to F, ϕ, b, d , if for all $x \in C$, there exists $\rho \in R$ such that

$$b(x, x_0)\phi[f(x) - f(x_0)] \leq 0 \Rightarrow$$

$$F(x, x_0; \xi) + \rho d^2(x, y) \leq 0, \forall \xi \in \partial^K f(x_0).$$

Definition 11 A function $f : C \rightarrow R$ is said to be weak uniform $K - (F_b, \rho)$ -quasiconvex at x_0 with respect to F, ϕ, b, d , if for all $x \in C$, there exists $\rho \in R$ such that

$$b(x, x_0)\phi[f(x) - f(x_0)] < 0 \Rightarrow$$

$$F(x, x_0; \xi) + \rho d^2(x, y) \leq 0, \forall \xi \in \partial^K f(x_0).$$

Sufficient optimality conditions

In this section, we obtain some sufficient conditions for a feasible \bar{x} to be efficient or weak efficient for (VFP) in the form of the following theorems.

Theorem 1 Assume that $\bar{x} \in X^0$, if for any $x \in X^0$, there exist $F, \phi, \phi_2, b_1, b_2, \rho_1^i \in R, \rho_2^j \in R, \lambda_i > 0$,

$i = 1, 2, \dots, p, \mu_j \in \Lambda, j \in I(\bar{x})$, such that

(i) $A_i(x) = f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})}g_i(x)$ is uniform $K - (F_{b_i}, \rho_1^i)$ -convex at $\bar{x}, i = 1, 2, \dots, p$;

(ii) $h(x, u^j)$ is uniform $K - (F_{b_j}, \rho_2^j)$ -convex at $\bar{x}, j \in I(\bar{x})$;

(iii) $0 \in \sum_{i=1}^p \lambda_i \partial^K f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \mu_j \partial^K h(\bar{x}, u^j), \forall u^j \in U^*$;

(iv) $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0, \phi_1(0) = 0, \alpha \leq 0 \Rightarrow \phi_2(\alpha) \leq 0, b_1(x, \bar{x}) > 0, b_2(x, \bar{x}) \geq 0$;

(v) $\sum_{i=1}^p \lambda_i \rho_1^i + \sum_{j \in I(\bar{x})} \mu_j \rho_2^j \geq 0$.

Then \bar{x} is an efficient solution for (VFP).

Proof. by hypothesis (iii), there exists $\xi_i \in \partial^K f_i(\bar{x}), i = 1, 2, \dots, p$ and $\eta_j \in \partial^K h(\bar{x}, u^j), j \in I(\bar{x})$ such that

$$\sum_{i=1}^p \lambda_i \xi_i + \sum_{j \in I(\bar{x})} \mu_j \eta_j = 0.$$

so

$$F(x, \bar{x}; \sum_{i=1}^p \lambda_i \xi_i + \sum_{j \in I(\bar{x})} \mu_j \eta_j) = F(x, \bar{x}; 0) = 0. \tag{1}$$

Suppose that \bar{x} is not an efficient solution for (VFP), then there exists $x \in X^0$ and at least one $i_0 \in \{1, 2, \dots, p\}$ such that

$$\frac{f_{i_0}(x)}{g_{i_0}(x)} < \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})},$$

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(\bar{x})}{g_i(\bar{x})}, i = 1, 2, \dots, p, i \neq i_0.$$

thus

$$A_{i_0}(x) = f_{i_0}(x) - \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})}g_{i_0}(x) < 0 = A_{i_0}(\bar{x}),$$

$$A_i(x) = f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})}g_i(x) \leq 0 = A_i(\bar{x}), i = 1, 2, \dots, p, i \neq i_0.$$

by hypothesis (iv), we have

$$b_1(x, \bar{x})\phi[A_{i_0}(x) - A_{i_0}(\bar{x})] < 0,$$

$$b_1(x, \bar{x})\phi[A_i(x) - A_i(\bar{x})] \leq 0, i = 1, 2, \dots, p, i \neq i_0.$$

by hypothesis (i), we get

$$F(x, \bar{x}; \xi_{i_0}) + \rho_1^{i_0} d^2(x, \bar{x}) < 0, \forall \xi_{i_0} \in \partial^K A_{i_0}(\bar{x}),$$

$$F(x, \bar{x}; \xi_i) + \rho_1^i d^2(x, \bar{x}) \leq 0, \forall \xi_i \in \partial^K A_i(\bar{x}), i = 1, 2, \dots, p, i \neq i_0.$$

since $\lambda_i > 0, i = 1, 2, \dots, p$, we have

$$\sum_{i=1}^p \lambda_i F(x, \bar{x}; \xi_i) + \sum_{i=1}^p \lambda_i \rho_1^i d^2(x, \bar{x}) < 0, \forall \xi_i \in \partial^K A_i(\bar{x}), i = 1, 2, \dots, p. \tag{2}$$

observe that $h(x, u^j) \leq 0 = h(\bar{x}, u^j), j \in I(\bar{x})$, we have

$$h(x, u^j) - h(\bar{x}, u^j) \leq 0, j \in I(\bar{x}).$$

by hypothesis (iv), we have

$$b_2(x, \bar{x})\phi_2[h(x, u^j) - h(\bar{x}, u^j)] \leq 0, j \in I(\bar{x}).$$

by hypothesis (ii), we get

$$F(x, \bar{x}; \eta_j) + \rho_2^j d^2(x, \bar{x}) \leq 0, \forall \eta_j \in \partial^K g(\bar{x}, u^j), j \in I(\bar{x}).$$

since $\mu_j \in \Lambda, j \in I(\bar{x})$, we have

$$\sum_{j \in I(\bar{x})} \mu_j F(x, \bar{x}; \eta_j) + \sum_{j \in I(\bar{x})} \mu_j \rho_2^j d^2(x, \bar{x}) \leq 0. \tag{3}$$

adding (1) and (3), using the sublinearity of F , we can obtain

$$F(x, \bar{x}; \sum_{i=1}^p \lambda_i \xi_i + \sum_{j \in I(\bar{x})} \mu_j \eta_j) + (\sum_{i=1}^p \lambda_i \rho_1^i + \sum_{j \in I(\bar{x})} \mu_j \rho_2^j) d^2(x, \bar{x}) < 0,$$

$$\forall \xi_i \in \partial^K A_i(\bar{x}), i = 1, 2, \dots, p, \forall \eta_j \in \partial^K g(\bar{x}, u^j), j \in I(\bar{x}).$$

by hypothesis (v), we have

$$\sum_{i=1}^p \lambda_i \rho_1^i + \sum_{j \in I(\bar{x})} \mu_j \rho_2^j \geq 0.$$

so

$$F(x, \bar{x}; \sum_{i=1}^p \lambda_i \xi_i + \sum_{j \in I(\bar{x})} \mu_j \eta_j) < 0,$$

$$\forall \xi_i \in \partial^K A_i(\bar{x}), i = 1, 2, \dots, p, \forall \eta_j \in \partial^K g(\bar{x}, u^j), j \in I(\bar{x}).$$

which contradicts (1). Therefore, \bar{x} is an efficient solution for (VFP).

Theorem 2 Assume that $\bar{x} \in X^0$, if for any $x \in X^0$, there exist $F, \phi_1, \phi_2, b_1, b_2, \rho_1^i \in R, \rho_2^j \in R, \lambda_i > 0, i = 1, 2, \dots, p, \mu_j \in \Lambda, j \in I(\bar{x})$, such that

(i) $A_i(x) = f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})} g_i(x)$ is uniform

$K - (F_{b_1}, \rho_1^i)$ -convex at $\bar{x}, i = 1, 2, \dots, p$;

(ii) $h(x, u^j)$ is uniform $K - (F_{b_2}, \rho_2^j)$ -convex at $\bar{x}, j \in I(\bar{x})$;

(iii) $0 \in \sum_{i=1}^p \lambda_i \partial^K f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \mu_j \partial^K h(\bar{x}, u^j), \forall u^j \in U^*$;

(iv) $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0, \phi_1(0) = 0, \alpha \leq 0 \Rightarrow \phi_2(\alpha) \leq 0, b_1(x, \bar{x}) > 0, b_2(x, \bar{x}) \geq 0$;

(v) $\sum_{i=1}^p \lambda_i \rho_1^i + \sum_{j \in I(\bar{x})} \mu_j \rho_2^j \geq 0$.

Then \bar{x} is a weak efficient solution for (VFP).

Theorem 3 Assume that $\bar{x} \in X^0$, if for any $x \in X^0$, there exist $F, \phi_1, \phi_2, b_1, b_2, \rho_1^i \in R, \rho_2^j \in R, \lambda_i > 0, i = 1, 2, \dots, p, \mu_j \in \Lambda, j \in I(\bar{x})$, such that

(i) $A_i(x) = f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})} g_i(x)$ is strictly uniform

$K - (F_{b_1}, \rho_1^i)$ -pseudoconvex at $\bar{x}, i = 1, 2, \dots, p$;

(ii) $h(x, u^j)$ is uniform $K - (F_{b_2}, \rho_2^j)$ -quasiconvex at $\bar{x}, j \in I(\bar{x})$;

(iii) $0 \in \sum_{i=1}^p \lambda_i \partial^K f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \mu_j \partial^K h(\bar{x}, u^j), \forall u^j \in U^*$;

(iv) $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0, \phi_1(0) = 0, \alpha \leq 0 \Rightarrow \phi_2(\alpha) \leq 0, b_1(x, \bar{x}) > 0, b_2(x, \bar{x}) \geq 0$;

(v) $\sum_{i=1}^p \lambda_i \rho_1^i + \sum_{j \in I(\bar{x})} \mu_j \rho_2^j \geq 0$.

Then \bar{x} is an efficient solution for (VFP).

Theorem 4 Assume that $\bar{x} \in X^0$, if for any $x \in X^0$, there exist $F, \phi_1, \phi_2, b_1, b_2, \rho_1^i \in R, \rho_2^j \in R, \lambda_i > 0, i = 1, 2, \dots, p, \mu_j \in \Lambda, j \in I(\bar{x})$, such that

(i) $A_i(x) = f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})} g_i(x)$ is uniform

$K - (F_{b_1}, \rho_1^i)$ -pseudoconvex at $\bar{x}, i = 1, 2, \dots, p$;

(ii) $h(x, u^j)$ is uniform $K - (F_{b_2}, \rho_2^j)$ -quasiconvex at $\bar{x}, j \in I(\bar{x})$;

(iii) $0 \in \sum_{i=1}^p \lambda_i \partial^K f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \mu_j \partial^K h(\bar{x}, u^j), \forall u^j \in U^*$;

(iv) $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0, \phi_1(0) = 0, \alpha \leq 0 \Rightarrow \phi_2(\alpha) \leq 0, b_1(x, \bar{x}) > 0, b_2(x, \bar{x}) \geq 0$;

(v) $\sum_{i=1}^p \lambda_i \rho_1^i + \sum_{j \in I(\bar{x})} \mu_j \rho_2^j \geq 0$.

Then \bar{x} is a weak efficient solution for (VFP).

Theorem 5 Assume that $\bar{x} \in X^0$, if for any $x \in X^0$, there exist $F, \phi_1, \phi_2, b_1, b_2, \rho_1^i \in R, \rho_2^j \in R, \lambda_i \geq 0, i = 1, 2, \dots, p, \mu_j \in \Lambda$, and not all μ_j is zero, $j \in I(\bar{x})$, such that

(i) $A_i(x) = f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})} g_i(x)$ is uniform

$K - (F_{b_1}, \rho_1^i)$ -quasiconvex at $\bar{x}, i = 1, 2, \dots, p$;

(ii) $h(x, u^j)$ is strictly uniform $K - (F_{b_2}, \rho_2^j)$ -pseudoconvex at $\bar{x}, j \in I(\bar{x})$;

(iii) $0 \in \sum_{i=1}^p \lambda_i \partial^K f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \mu_j \partial^K h(\bar{x}, u^j), \forall u^j \in U^*$;

(iv) $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0, \phi_1(0) = 0, \alpha \leq 0 \Rightarrow \phi_2(\alpha) \leq 0, b_1(x, \bar{x}) \geq 0, b_2(x, \bar{x}) \geq 0$;

(v) $\sum_{i=1}^p \lambda_i \rho_1^i + \sum_{j \in I(\bar{x})} \mu_j \rho_2^j \geq 0$.

Then \bar{x} is an efficient solution for (VFP).

Theorem 6 Assume that $\bar{x} \in X^0$, if for any $x \in X^0$, there exist $F, \phi_1, \phi_2, b_1, b_2, \rho_1^i \in R, \rho_2^j \in R, \lambda_i \geq 0, i = 1, 2, \dots, p, \mu_j \in \Lambda$, and not all μ_j is zero, $j \in I(\bar{x})$, such that

(i) $A_i(x) = f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})} g_i(x)$ is weak uniform

$K - (F_{b_1}, \rho_1^i)$ -quasiconvex at $\bar{x}, i = 1, 2, \dots, p$;

(ii) $h(x, u^j)$ is strictly uniform $K - (F_{b_2}, \rho_2^j)$ -pseudoconvex at $\bar{x}, j \in I(\bar{x})$;

(iii) $0 \in \sum_{i=1}^p \lambda_i \partial^K f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \mu_j \partial^K h(\bar{x}, u^j), \forall u^j \in U^*$;

(iv) $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0, \phi_1(0) = 0, \alpha \leq 0 \Rightarrow \phi_2(\alpha) \leq 0, b_1(x, \bar{x}) \geq 0, b_2(x, \bar{x}) \geq 0$;

(v) $\sum_{i=1}^p \lambda_i \rho_1^i + \sum_{j \in I(\bar{x})} \mu_j \rho_2^j \geq 0$.

Then \bar{x} is a weak efficient solution for (VFP).

The proofs of Theorem 2—Theorem 6 are similar to Theorem 1.

Conclusions

In this paper, we consider nonsmooth multi-objective fractional semi-infinite programming involving a new classes of generalized convex functions, that is, uniform $K - (F_b, \rho)$ -convex function, uniform $K - (F_b, \rho)$ -pseudoconvex function, uniform $K - (F_b, \rho)$ -quasiconvex function and obtain some sufficient optimality conditions.

Acknowledgements

This work was supported by scientific research fund of Shaanxi province education department(14JK1862). The author thank the referees for many valuable comments and helpful suggestions.

References

1. Hanson M.A., Mond M. (1987) Necessary and Sufficient Conditions in Constrained Optimization. *Mathematical Programming*, 37(3), p.p.51-58.
2. Rueda N.G., Hanson M.A.(1998)Optimality Criteria in Mathimatical Programming Involving Genelized Invexity. *Journal of Mathematical Analysis and Applications*,130 (6), p.p.375-385.
3. Bector C. R., Singh C. (1991) B-Vex Function. *Journal of Optimization Theory and Applications*, 71(5), p.p.237-253.
4. Bector C.R, Suneja S.K., Gupta S. (1992) Univex Functions and Univex Nonlinear Programming. *Proceedings of the Administrative Science Association of Canada*, Vancouver, Canada, p.p.115-124.
5. Mishra S.K. (1998) On Multiple-Objective Optimization with Generalized Univexity. *Journal of Mathematical Analysis and Applications*, 224(11), p.p.131-148.
6. V.Preda.(1992) On Efficiency and Duality for Multi-objective Programs. *Journal of Mathematical Analysis and Applications*, 166(8), p.p.365-377.
7. Hanson M.A., Mond M. (1986) Further Generalizations of Convexity in Mathematical Programming. *Journal of Information and Optimal Science*, 22(3),25-32.
8. J.P.Vial.(1989) Strong and Weak Convexity and Duality in Multiple Objective Programming.
9. *Bulletin of the Australia Mathematical Society*, 39(2), p.p.287-299.
10. Hong Yang (2014) Duality for multi-objective simi-infinite programming with $K - (F_b, \rho)$ -convexity. *Computer Modelling and New Technologies*, 18(11), p.p.574-577.
11. Hong Yang.(2014) Sufficiency of the solution for multi-objective simi-infinite programming with $K - (F_b, \rho)$ -convexity. *Computer Modelling and New Technologies*, 18(12A), p.p.270-274.
12. BectorC. R.,S.Chandra, I.Husian (1993) Optimality Conditions and Duality in Subdifferentiable Multiobjective Fractional Programming. *Journal of Optimization Theory and Applications*, 79(1), p.p.105-125.
13. Liu J.C.,Y.Kimura, K.Tanaka (1999) Three Types Dual Model for Minimax Fractional Programming. *Computers & Mathematics with Applications*, 38(2), p.p.143-155.
14. H.Kuk, G.M.Lee, T.Tanino (2001) Optimality and Duality for Nonsmooth Multiobjective Fractional Programming with Generalized Invexity. *Journal of Mathematical Analysis and Applications*, 262(3), p.p.365-375.
15. S.K.Mishra, R.P.Pant, J.S.Rautela (2009) Generalized α -univexity and Duality for Nondifferentiable Minimax Fractional Programming. *Nonlinear Analysis*, 70(1), p.p.144-158.
16. Shunchin Ho, Hangchin Lai.(2012) Optimality and Duality for Nonsmooth Minimax Fractional Programming Problem with Exponential (p, r) -Invexity. *Journal of Nonlinear and Convex Analysis*, 65(3), p.p.433-447.
17. A.K.Tripathy (2014) Mixed Type Duality in Multiobjective Fractional Programming under Generalized ρ -univex Function. *Journal of Mathematical Model and Algorithm*,13(1), p.p.331-340.
18. K.H.Elster, Thierfelder (1988) Abstract Cone Approximations and Generalized Differentiability in Nonsmooth Optimization. *Optimization*,19(3), p.p.315-341.