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Method of boundary elements in problems of stability of plane bending of rectangular section beams

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Abstract
An algorithm for solving problems of stability of plane bending of beams of rectangular section (thin strips) with the help of numerical and analytical variant of the method of boundary elements. The aim is to develop new solutions of differential equations of stability problems. Beams with sections in the form of narrow strips have higher strength and stiffness, however, in case of the transverse load; there is a risk of loss of plane bending stability. In this case, the beam is further bent in another plane and twists. There emerge flexural-torsional buckling, in which large displacements and structural failure may occur. The theory of solving such problems needs to be developed, as current results are very difficult to extend to continuous beams and frames. The boundary element method can greatly simplify the process of solution to improve the accuracy and reliability of the results obtained and disseminate solutions to more complex structure than a beam. Calculations of the critical forces in are conducted MATLAB medium.
Key words: BOUNDARY ELEMENT METHOD, THE STABILITY OF PLANE BENDING, RECTANGULAR BEAM, MATLAB

Introduction

Beams with sections in the form of narrow strips have higher strength and stiffness. In this regard, they are of great use in a variety of beam and frame structures of engineering, construction, bridges, etc. However, with a large ratio of height to width of the section, there is a real danger of buckling plane bending of such beams. It further obtains the troughs in the other plane and the angles of twist. If the displacement is too large, the structure collapses. Therefore, it is important to have a reliable, accurate and relatively simple theory of solving such problems of sustainability.

Analysis of recent research and publications

The first solution of problems of stability of plane bending beams with a cross section in the form of narrow strips was obtained in the 19th century [1]. To date, quite a lot of problems of this type are solved [2, 3]. In solving problems of stability of beams under lateral load differential equation; and its solution can be written for the angle of twist. The functions representing an alternating infinite power series where used in solution. In this case, accuracy of the result depends on the number of terms that are not always convenient. In addition, existing solutions are very difficult to be used to solving problems of stability of continuous beams and frames, as they are incomplete (does not include other options). If we apply the algorithm of numerically-analytical boundary element method (MBE) to these problems [4], it will be possible to simplify the procedure solutions to increase the accuracy and reliability of the results and apply new solutions of differential equations in more complex structures than simple beams.

Objective. The aim is to build new solutions of differential equations of stability problems of plane bending beams with a cross section in the form of a narrow band, and the use of these solutions for specific problems.Statement of the base material.

The differential equation of the stability of the I-beam was derived by S. P. Timoshenko [1]

$$\begin{cases} EI_y w^{IV}(x) + M_Z(x)\theta^{II}(x) = 0; \\ EI_\omega \theta^{IV}(x) - GI_d \theta^{II}(x) + M_z(x) w^{II}(x) = 0, \end{cases} \quad (1)$$

where

$w(x)$ – displacement of the rod axis in the direction of the axis Oz;

$\theta(x)$ – the twist angle of the rod axis Ox;

EI_y – the flexural rigidity of the rod in a plane xOz;

EI_w – sectorial stiffness;

GI_d – torsional rigidity;

$M_Z(x)$ – bending moment in the cross section of the rod with respect to the axis Oz caused by a given lateral load (Figure1)

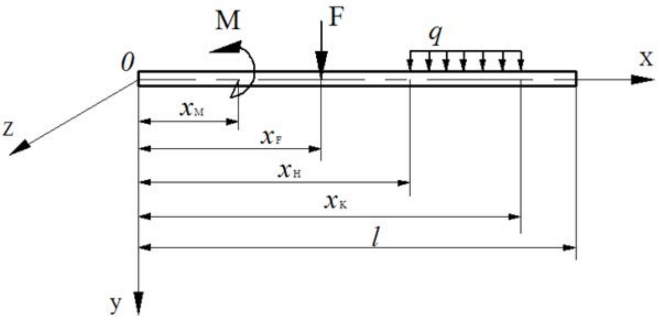

Figure 1. The core of constant cross section in a single-rotor coordinate system

Since the rectangular section also has two axis symmetry, the system of equations (1) can be used for this case with a substantial simplification. Thin rectangular torsion has a very small value and warping member $EI_\omega \theta^{IV}(x)$, which can be neglected. The equations of the stability of rod with rectangular cross section can be written as

$$\begin{cases} EI_y w^{IV}(x) + M_Z(x)\theta^{II}(x) = 0; \\ -GI_d \theta^{II}(x) + M_z(x) w^{II}(x) = 0, \end{cases} \quad (2)$$

From the last equation of the system (2) the relationship between the angle of twist and deflection rod is

$$-GI_d\theta(x) + M_z(x)w(x) = Ax + B, \quad (3)$$

where the constants of integration are determined from the initial conditions

$$\begin{aligned} B &= -GI_d\theta(0) + M_z(0)\cdot w(0); \\ A &= -GI_d\theta'(0) + M_z(0)\cdot w'(0). \end{aligned} \quad (4)$$

From the second equation of (2), it follows that

$$\theta''(x) = \frac{M_z(x)}{GI_d} w'(x), \quad (5)$$

Substituting (5) into the first equation (2), we obtain the equation of stability of rod of rectangular cross section

$$w^{IV}(x) + \frac{M_z^2(x)}{GI_d EI_y}\cdot w''(x) = 0. \quad (6)$$

This equation is an ordinary differential equation with variable coefficients. Its integration leads to significant mathematical difficulties, especially when a bending moment $M_Z(x)$ has several laws of change, the points of 1-order discontinuity and the break point. All these difficulties are overcome, if the solution of problems of stability of the algorithm MBE is applied [4]. To this end, we represent the solution of the equation (6) in the case where the stiffness GI_d, EI_y and bending moment M_z are constant along the length of the rod. It can be seen that equation (6) in the form of recording is different from the equation of longitudinal-transverse bending the straight rod, in case when the longitudinal force is compressive [5]. Its matrix solution is of the form

		1	2	3	4		
$EI_y w(x)$	1	1	x	-A_{13}	-A_{14}		$EI_y w(0)$
$EI_y \varphi(x)$	= 2		1	-A_{23}	-A_{13}		$EI_y \varphi(0)$
$M_y(x)$	3			A_{33}	A_{23}		$M_y(0)$
$Q_z(x)$	4			A_{43}	A_{33}		$Q_z(0)$

(7)

where $EI_y\varphi(x), M_y(x), Q_z(x)$ – options of bending rod horizontal xOz. Orthogonal fundamental functions are of the form

$$A_{13} = \frac{1-\cos nx}{n^2}; \quad A_{14} = \frac{nx - \sin nx}{n^3};$$
$$A_{23} = \frac{\sin nx}{n}; \quad A_{33} = \cos nx; \qquad (8)$$
$$A_{43} = -n \sin nx; \quad n = \sqrt{\frac{M_z^2}{GI_d EI_y}}$$

From the presented solution, it follows that the problems of stability of a rod with rectangular cross section are described by the relatively simple trigonometric functions. Options torsion in the solution (7) is not involved, but there is a connection between bending and torsion in the form of (3). When forming the boundary value problems, the boundary conditions of the beam in the horizontal plane xOz are used. If M_z is a function of $M_z = M_z(x)$, the moment diagram is divided into *m* parts, within which $M_{zi} = const$ and the solution (7) of the MBE algorithm can be used for various tasks sustainability. In case of sufficiently large *m*, the results close to accurate are obtained. Let us consider some stability problems of beams.

Results. Beams with constant section in which $M_z = const$. These beams include constructions, in which the boundary points have hinged support in the vertical plane xOy loaded with concentrated moments at the ends. The horizontal xOz may have different support conditions. Transverse load is considered to be applied to the longitudinal axis of the rods. The bending moment is convenient to be represented by the expression

$$M_z = a_M \cdot M + a_F \cdot F + a_q \cdot q, \qquad (9)$$

where a_M, a_F, a_q – bending moments caused by an appropriate lateral load equal to unity; *M, F, q* – the critical values of the lateral load. In order to determine the critical load values, it is necessary to make the equation of boundary value problem and find the roots of the transcendental equation MBE [4,5]

$$|A_*(M,F,q,l)| = 0 \qquad (10)$$

where $A_*(M,F,q,l)$ – the equation (7) matrix converted by an algorithm MBE coefficient. Let form the matrix for a number of tasks.

Object 1. The beam has a swivel bearing in two planes. If the boundary value of *x* = *l* we have

A_*

	1	2	3	4
1		1		-A_{14}
2	-1	1		-A_{13}
3				A_{23}
4			-1	A_{33}

X_*

$EI_y\omega(0)=0; EI_y\varphi(l)$
$EI_y\varphi(0)$
$M_y(0)=0; Q_z(l)$
$Q_z(0)$

–

Y

$EI_y\omega(l)=0$
$EI_y\varphi(l)$
$M_y(l)=0$
$Q_z(l)$

=0 (11)

From the initial parameters of the matrix X_*, it follows that the matrix A_* columns 1 and 3 must be reset, as they are associated with zero parameters. Non-zero parameters of the matrix Y are transferred to their place, which leads to compensating elements $A_*(2,1) = -1$; $A_*(4,3) = -1$ in matrix A_*. Expansion elements are the coefficients of the portable options and must be in the matrix A_* in place (*i, j*), where *i* – old address (line number matrix Y), *j* – new address (the number of rows of the matrix X_*) of parameter. In other words, the chain of elementary matrix transformations is fulfilled

$$Y(l) = A(l)\cdot X(0) \rightarrow A(l)\cdot X - Y(l) = 0 \rightarrow A_*(l)\cdot X_*(0,l) = 0 \qquad (12)$$

The last equality implies BEM equation for determination of critical values of the lateral load (10). For this task, the roots of (10) can be determined analytically. The determinant of the matrix $A_*(M,l)$ equals

$$\left|A_*(M,l)\right| = l\cdot A_{23} + 0\cdot A_{14} = 0. \qquad (13)$$

If we substitute the values of the fundamental function A_{23} into equation (13), we obtain the equation

$$\sin nl = 0 \rightarrow M_1 = \frac{\pi}{l}\sqrt{GI_d\cdot EI_y}, \qquad (14)$$

that is an exact solution [2, 3].

Object 2. The beam has a tight seal in the horizontal plane. The determinant of the equation (10) in this case is of the form

$$\left|A_*(M,l)\right| = A_{13}^2 - A_{14}\cdot A_{23} = 0. \qquad (15)$$

The root of this equation must be sought through a computer. To solve this and other problems, the following inputs are accepted:

$$E = 2\cdot 10^8 кН/м^2;\ G = 0{,}8\cdot 10^8 кН/м^2;$$

$$b = 0{,}01м;\ h = 0{,}1м;$$

$$I_y = hb^3/12 = 8{,}333\cdot 10^{-9} м^4;$$

$$I_d = \beta hb^3 = 0{,}313\cdot 0{,}1\cdot 0{,}01^3 = 31{,}3\cdot 10^{-9} м^4;$$

$$EI_y = 1{,}666 кНм^2;\ GI_d = 2{,}504 кНм^2;\ l = 1{,}0\ м.$$

After setting the data in the program, the first root (critical value) of the equation (15) is

$$M_1 = 12{,}841\ кНм;$$

or in the dimensionless form

$$M_1 = \frac{2\pi}{l}\sqrt{GI_d\cdot EI_y}, \qquad (16)$$

which is also the exact solution [2,3].

Object 3. The beam has tough termination and swivel bearing in the horizontal plane. The determinant of the equation (10) is of the form

$$\left|A_*(M,l)\right| = -A_{13}\cdot A_{23} + A_{14}\cdot A_{33} = 0. \qquad (17)$$

The first root is equal to

$$M_1 = 9{,}175кНм \rightarrow M_1 = \frac{4{,}491}{l}\sqrt{GI_d\cdot EI_y} \qquad (18)$$

this result is not presented in the reference data [2,3].

Object 4. Cantilever beam is loaded by concentrated moment at the end. In this case

$$\left|A_*(M,l)\right| = A_{33}^2 - A_{23}\cdot A_{43} = 0. \qquad (19)$$

If we substitute the values of the fundamental functions into this equation, we obtain the unsatisfiable equality

$$\cos^2 nx + \sin^2 nx = 1 = 0. \qquad (20)$$

This suggests that an incompletely stability equation (7) does not allow determining the critical load. It is necessary to supplement it with the parameters torsion. The differential equation of stability for the angle of twist is of the form [2]

$$\theta'' + \frac{M_z^2(x)}{GI_d\cdot EI_y}\theta(x) = 0. \qquad (21)$$

According to the notation, equation does not different from the equation of torsional vibrations of the prismatic core. Therefore, its matrix solution can be written as [5]

$$\begin{vmatrix} GI_d\theta(x) \\ M_{кр}(x) \end{vmatrix} = \begin{vmatrix} \cos nx & \frac{\sin nx}{n} \\ -n\sin nx & \cos nx \end{vmatrix} \begin{vmatrix} GI_d\theta(0) \\ M_{кр}(0) \end{vmatrix} \qquad (22)$$

If we apply the quasi diagonalization operation, the complete solution of the stability of the rod of rectangular cross section will be as follows:

		1	2	3	4	5	6		
$EI_y w(x)$	1	1	x	$-A_{13}$	$-A_{14}$				$EI_y\omega(0)$
$EI_y\varphi(x)$	2		1	$-A_{23}$	$-A_{13}$				$EI_y\varphi(0)$
$M_y(x)$	= 3			A_{33}	A_{23}				$M_y(0)$
$Q_z(x)$	4			A_{43}	A_{33}				$Q_z(0)$
$GI_d\theta(x)$	5					A_{33}	A_{23}		$GI_d\theta(0)$
$M_{кр}(x)$	6					A_{43}	A_{33}		$M_{кр}(0)$

(23)

Let us apply this equation to this problem and take into account the boundary conditions.

The matrix coefficients $A_*(M,l)$ takes the form

	①	②	3	4	⑤	6			
1	-1		$-A_{13}$	$-A_{14}$			$EI_y w(0)=0; EI_y w(l)$	←	$EI_y w(l)$
2		-1	$-A_{23}$	$-A_{13}$			$EI_y \varphi(0)=0; EI_y \varphi(l)$	←	$EI_y \varphi(l)$
3			A_{33}	A_{23}			$M_y(0)$		$M_y(l)=0$
4			A_{43}	A_{33}			$Q_z(0)$	–	$Q_z(l)=0$
5					-1	A_{23}	$GI_d \theta(0)=0; GI_d \theta(l)$	←	$GI_d \theta(l)$
6						A_{33}	$M_{xp}(0)$		$M_{xp}(l)=0$

$= 0$ (24)

The equation of the beam stability

$$A_*(M,l) = (A_{33}^2 - A_{23} \cdot A_{43}) \cdot A_{33} = 0 \quad (25)$$

has the first root

$$M_1 = \frac{\pi}{2l}\sqrt{GI_d \cdot EI_y}, \quad (26)$$

that is the exact result [2,3].

Beams with a constant cross-section and $M_z = M_z(x)$. This case is widespread in practice. However, the differential equation (6) in this case is very difficult to be integrated. It is suggested to solve boundary task for the equation with variable coefficient using fundamental functions of the equation, but with a constant factor. To this end, the beam is sampled into m parts. In each of the calculated optimum constant value M_{zi}. Next to their sites M_{zi} joined by the rules of the MBE. For example, when braking down of beams with pivot bearing into 3 parts will be of the form

	$X_*=$		$Y=$
1	$w_{(0)}^{0-1}=0;\ \varphi_{(l)}^{2-3}$	1	$w_{(l)}^{0-1}=w_{(0)}^{1-2}$
2	$\varphi_{(0)}^{0-1}$	2	$\varphi_{(l)}^{0-1}=\varphi_{(0)}^{1-2}$
3	$M_{y(0)}^{0-1}=0; Q_z^{2-3}(l)$	3	$M_y^{0-1}(l)=M_{(0)}^{1-2}$
4	$Q_z^{0-1}(0)$	4	$Q_z^{0-1}(l)=Q_z^{1-2}(0)$
5	$\omega_{(0)}^{1-2}$	5	$w_{(l)}^{1-2}=w_{(0)}^{2-3}$
6	$\varphi_{(0)}^{1-2}$	6	$\varphi_{(l)}^{1-2}=\varphi_{(0)}^{2-3}$
7	$M_{y(0)}^{1-2}$	7	$M_y^{1-2}(l)=M_{(0)}^{2-3}$
8	$Q_{z(0)}^{1-2}$	8	$Q_z^{1-2}(l)=Q_{(0)}^{2-3}$
9	$w_{(0)}^{2-3}$	9	$w_{(l)}^{2-3}=0$
10	$\varphi_{(0)}^{2-3}$	10	$\varphi^{2-3}(l)$
11	$M_{y(0)}^{2-3}$	11	$M_{(l)}^{2-3}=0$
12	$Q_{z(0)}^{2-3}$	12	$Q_z^{2-3}(l)$

(27)

From matrix X_* it follows that the matrix A_* the columns 1 and 3 must be reset, and the compensating elements A(10,1)= -1; A (12,3) = -1must be introduced.

Other compensation elements related to the equations of kinematic and static parameters in the interior of the beam form a diagonal from (-1) in the appropriate place. In this case, the matrix equation for the critical load (10) takes the form

$A_*=$

	1	2	3	4	5	6	7	8	9	10	11	12
1		l^{0-1}		$-A_{14}/EI_y$	-1							
2		1		$-A_{13}/EI_y$		-1						
3				A_{23}			-1					
4				A_{33}				-1				
5					1	l^{1-2}	$-A_{13}/EI_y$	$-A_{14}/EI_y$	-1			
6						1	$-A_{23}/EI_y$	$-A_{13}/EI_y$		-1		
7							A_{33}	A_{23}			-1	
8							A_{43}	A_{33}				-1
9									1	l^{2-3}	$-A_{13}/EI_y$	$-A_{14}/EI_y$
10	-1									1	$-A_{23}/EI_y$	$-A_{13}/EI_y$
11											A_{33}	A_{23}
12			-1								A_{43}	A_{33}

(28)

To simplify the equations docking sites (see the matrix Y (27)), stiffness EIy is introduced to the matrix A_*. A small number of sections of the beam results in rough approximation of the function $M_z(x)$ and the critical values are inaccurate. In case of increasing the number of sections m, its boundary value at which the critical forces are not specified, can be determined. At that, matrix A_* is of corresponding size. Let us consider a test case.

Object 5. The beam loaded with lumped force F with simple support in two planes. The matrix A_* (28) at $m = 50$ is used. Results of the solution are summarized in Table 1.

Table 1. Critical power when changing coordinates of forces F

Coordinate of force F	0,1*l*	0,2*l*	0,3*l*	0,4*l*	0,5*l*
MBE	56,03	29,12	21,01	17,83	16,95
Data [2]	56,01	29,11	21,01	17,82	16,94
Measure of inaccuracy,%	0,04	0,03	0,0	0,06	0,06

Object 6. The beam loaded with concentrated force *F* with simple support in the vertical plane xOy and seals in horizontal plane xOz. Columns 1, 2 and 5 must be reset in matrix A_* (28), and the offsetting items must be introduced. Calculations of critical forces show that at $m \geq 30$ their specifications are almost finished. The results of this task at $m = 50$ and complete solution (23) are summarized in Table 2

Table 2. The critical force of task 6

Coordinate of force F	0,1*l*	0,2*l*	0,3*l*	0,4*l*	0,5*l*
MBE	88,45	45,96	33,21	28,20	26,79
Date [2]	177	53,2	35,2	28,5	26,7
Measure of inaccuracy,%	24,40	13,60	5,66	1,07	0,32

Object 7. The beam loaded with uniformly distributed load and under various boundary conditions. Applying the proposed algorithm, the results of the solution are summarized in Table 3.

Table 3. Critical force under the effect of uniformly distributed load

Boundary conditions of support	MBE	Date [2]	The error results %
Hinged support in two planes	28,32	28,31	0,03
Hinged support in the plane xOy; hinge support and sealing in the plane xOz	39,77	39,6	0,44

Hinged support in the plane xOy; seal in the plane xOz	44,79	48,6	7,84

Object 8. The beam loaded with concentrated moment with various boundary conditions. The results of solving the task of stability of beams covering the possible support conditions under the action of distributed load along the length and concentrated forces are presented in reference data [2, 3]. There are no tasks when the concentrated time is in the beams span. It is necessary to take into account the gap of 1-order function of the bending moment $M_z(x)$. Apparently, considering this factor, the difficulties influenced the absence of these solutions. These difficulties are removed in the algorithm MBE. In the Table 4, the results for critical values of concentrated moments at $m = 50$ are presented.

Table 4. Critical moments of the beams loaded with concentrated moments

Boundary conditions of support	Coordinate of concentrated moment M									
	$0{,}0l$	$0{,}1l$	$0{,}2l$	$0{,}3l$	$0{,}4l$	$0{,}5l$	$0{,}6l$	$0{,}7l$	$0{,}8l$	$0{,}9l$
Hinged support in two planes	5,56	5,65	6,09	6,88	7,91	8,49	7,91	6,88	6,09	5,65
Hinged support in the plane *xOy*; hinge support and sealing in the plane *xOz*	6,51	6,83	7,51	8,88	10,70	11,72	10,99	10,21	9,57	8,93
Hinged support in the plane *xOy*; seal in the plane *xOz*	8,79	8,93	9,57	10,84	12,46	13,39	12,46	10,84	9,57	8,93

We should make a point that the data in Table 4 approximately reflect the critical value of moments. This is due to the fact that the differential equations (2) and their solutions (7), (22), (23) are insensitive to the sign of the bending moment $M_z(x)$. Therefore, the problem8 is solved for positive bending moment M_z. The exact solution can be obtained by using the differential equation (1) and the corresponding fundamental orthonormal functions system [4], which takes into account just sign $M_z(x)$.

Object 9. The cantilever beam has a section height changing according to the law

$$h = h_0(1 - x/l)^{1/\kappa} \qquad (29)$$

The beam is also sampled into m parts. For each part, bending moment M_{zi} is calculated when distributed load q at coordinate $x = x_i/\sqrt{3}$, where x_i – the length of the i-th section of the beam in case of concentrated force $x = x_i/2$ [4]. Flexural rigidity EI_{yi} and torsion GI_{di} were calculated in the middle of each section. The rest of the algorithm remains unchanged. Results of the solution at $m = 0$ are shown in Table 5.

Table 5. Critical loads cantilever beam with variable height

The method of loading		κ			
		4	2	1,333	1
Uniform load q	MBE	18,14	15,81	13,68	11,43
	Data [2]	12,8	11,2	10,4	9,6
	Measure of inaccuracy, %	41,72	41,16	31,54	19,06
concentrated force F at the end	MBE	5,26	4,13	3,01	1,93
	Data [2]	3,61	3,21	2,81	2,40
	Measure of inaccuracy, %	45,71	28,66	7,12	19,58

By way of illustration and comparison with the data of the Table 5, Table 6 shows critical loads of similar beam according to hinged support in two planes.

Table 6. Critical loads of supported beam of variable height

The method of loading	к			
	4	2	1,333	1
Uniformly distributed load *q*	23,32	18,82	12,12	6,34
Concentrated force *F* in the middle of span	14,07	11,48	8,40	4,92

Conclusions

Presented problems of stability of plane bending beams with a cross section in the form of a narrow strip show that the model (7), (23) and an algorithm for the numerical-analytic variant MBE provides accurate and reliable results in the constant and variable values of bending moment M_z and rigidities EI_y, GI_d. Arbitrary external load, a lot of laws of the time change M_z, break points and 1-order discontinuity may be considered. There are no difficulties when applying the suggested approach for solving of stability tasks of more complex constructions such as continuous beams and frames [4].

References

1. V. Z. Vlasov *Tonkostennye uprugie sterzhni*. [Thin elastic rods]. Moscow, L Fizmatgiz, 1959. 568 p.
2. Birger I.A., Panovno Y.G. *Prochnost', ustoychivost', kolebaniya.* [The strength, stability, vibrations. Reference in 3 volumes]. Moscow, Mechanical Engineering, 1968, 569 p.
3. Smirnov A.F. *Stroitel'naya mekhanika. Dinamika i ustoychivost' sooruzheniy*. [Structural mechanics. Dynamics and stability of buildings], Moscow, Stroyizdat, 1984. 416 p.
4. Dashchenko A.F. *Chislenno-analiticheskiy metod granichnykh elementov*. [Numerical-analytical boundary element method], Odessa, WWII, 2010. Vol.1, 416 p.
5. Bazhenov V.A. *Stroitel'naya mekhanika. Spetsial'nyy kurs. Primenenie MGE.* [Structural mechanics. Special course. The use of MBE]. Odessa, Astroprint, 2001. 288 p.