Evolutionary game model with time delay

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Abstract
To study the interactions and learning effect between players, this paper constructs two evolutionary game dynamic systems. One is with time delay, the other is without time delay. The paper analyzes the stabilities of equilibrium points in two evolutionary game systems on theory. We find that the stabilities of the 5 equilibrium points are different though two evolutionary game systems have the same equilibrium points. Though time delay has no impact on the situation of the final stable state, it may change the speed of system convergence to the stable points. The numerical simulations of this paper have verified that time delay and initial conditions have influence on the speed of convergence. So the conclusions of this paper mean that players will make more reasonable decisions if they make full use of the present and pass information in the evolutionary game.
Key words: TIME DELAY, EVOLUTIONARY GAME, NUMERICAL SIMULATION, DYNAMIC SYSTEM.

1. Introduction

Since evolutionary game theory was used in selection among competing strategies by Axelrod [1], it has drawn a highly attention of many social scientists [2]. And then evolutionary game theory is widely used in not only explaining the competition of species but also modelling how social phenomena can come into being. Under the assumption that utility-maximizing individuals interact with each other, evolutionary game models account for various phenomena, for example, the emergence of altruism [3, 4, 5], social learning [6, 7], social norms [8, 9], and moral behavior [10, 11].

Two approaches of evolutionary game theory are commonly applied in the research. One approach concerns the equilibrium concepts. Among them, the equilibrium concepts are regarded as the initial tool of analysis. For example, it is common to prove and analyze stability characteristics of ESS in order to show the static outcome of evolutionary game. The other approach emphasizes the dynamics of evolutionary game. So the key issue is to construct an evolutionary game model with various frequencies of strategies and study its' evolutionary dynamics. In most studies on evolutionary game models have two implications. One is that players have bounded rationality. The other is that player inherit their strategies. These implications mean that players are from sizeable populations and they are not inclined to influence other player's future actions. However, it is common in the real market that players have a learning effect during the game, which means players can affect each other. In recent years, there has been some other works on the dynamical game with bounded rationality. The models with time delay are used in dynamical Cournot game [12, 13].

Though evolutionary game is a kind of dynamical game, few works consider the evolutionary games with time delay. So we combine the evolutionary game model with time delay and analyze its' equilibrium in order to capture the interactions between players. The content of this paper is organized as the following: In section 2, we model an evolutionary game with two players. In section 3, the stabilities of evolutionary game models with and without time delay are discussed. Section 4 gives the dynamic features of the systems with and without time delay by the numerical simulations. Some conclusions are proposed in the last section.

2. The model

2.1 The basic model

We consider a competition between two players, labeled by A and B. two players are not with perfect rationality. The strategy of each player is to cooperate or to defect. We also suppose that the players are heterogeneity. The payoff matric is shown in Table 1.

Table 1. The payoff matric of players

	Player B cooperates	Player B defects
Player A cooperates	R , S	E , F
Player A defects	C , D	G , H

Because of the heterogeneity, the payoffs for the player A and player B are asymmetric. If they all cooperate, the reward for player A is R while S for player B. The punishments for player A and B are G and H if they all defect. If player A defects while player B cooperates, the payoffs for player A and B are C and D. If player A cooperates while player B defects, the payoffs for player A and B are E and F.

Following others in the evolutionary game model, we suppose that x represents the possibility of player A to cooperate. Then $(1-x)$ corresponds to the possibility to defect for player A. we use y to denote the possibility for player B to cooperate. So $(1-y)$ means the possibility to defect for player B.

According to the game theory, when the choice for player A is to cooperate, his or her expected payoff is then equal to:

$$U_C^A = R\cdot y + E\cdot(1-y). \quad (1)$$

And if player A chooses defection, his or her expected payoff is given by:

$$U_D^A = C\cdot y + G\cdot(1-y). \quad (2)$$

In light of the equation (1) and (2), the mean return of player A is computed as:

$$\bar{U}^A = x\cdot U_C^A + (1-x)\cdot U_D^A = (R-E-C+G)xy + (E-G)x + (C-G)y + G \quad (3)$$

We suppose the possibility for player A to cooperate is increasing as time goes on. So x increases according to the replicator dynamic equation as following：

$$\begin{aligned}\frac{dx}{dt} &= x\left(U_C^A - \bar{U}^A\right) = x(1-x)\left(U_C^A - U_D^A\right) \\ &= x(1-x)\left((R-E-C+G)y + E - G\right)\end{aligned} \quad (4)$$

Similarly, the replicator dynamic equation for player A's possibility to cooperate reads:

$$\begin{aligned}\frac{dy}{dt} &= y\left(U_C^B - \bar{U}^B\right) = y(1-y)\left(U_C^B - U_D^B\right) \\ &= y(1-y)\left((S-D-F+H)x + D - H\right)\end{aligned} \quad (5)$$

So the evolutionary game model for player A and player B can be regarded as a dynamical system as follows:

$$\begin{cases}\dfrac{dx}{dt} = x(1-x)\left((R-E-C+G)y + E - G\right) \\ \dfrac{dy}{dt} = y(1-y)\left((S-D-F+H)x + D - H\right)\end{cases} \quad (6)$$

2.2 Time delay

The model (6) implies that the next decisions of players in the game are only related to the current decisions. But this implication may not keep pace with our life. As we all known, the past information is as valuable as the present information in the decision. A common way in dynamic system is used the time delays to embody the past information. Therefore, we adjust the model (6) by introducing a time delay. For simplicity, we consider an evolutionary game with one order time delay.

$$\begin{cases}\dfrac{dx}{dt} = x(1-x)\left((R-E-C+G)y(t-\tau_1) + E - G\right) \\ \dfrac{dy}{dt} = y(1-y)\left((S-D-F+H)x(t-\tau_2) + D - H\right)\end{cases} \quad (7)$$

So the model (7) is a dynamic system with a time delay.

In order to study the stability of model (6) and (7), we suppose $\frac{dx}{dt} = \frac{dy}{dt} = 0$. And then we obtain five nonnegative equilibrium points for model (6).

$E_1 = (0,0)$, $E_2 = (0,1)$, $E_3 = (1,0)$, $E_4 = (1,1)$,

$$E_5 = \left(\frac{D-H}{M}, \frac{E-G}{N}\right).$$

In the above equilibrium points, M is equal to $(S-D-F+H)$, N denotes $(R-E-C+G)$. And we suppose $M, N > 0$. The five equilibrium points of model (7) are the same as model (6). Among the five equilibrium points, only E_5 is an interior equilibrium and the other four points are all boundary equilibriums.

3. Stability of equilibrium

3.1 Stability in system (5)

To study the local stability of an equilibrium point $E=(x,y)$ of system (5), we work out the eigenvalues of the Jacobian matrix J, which takes the form as:

$$J(x,y)=\begin{pmatrix}(Ny+E-G)(1-2x) & x(1-x)N \\ y(1-y)M & (Mx+D-H)(1-2y)\end{pmatrix}$$

(8)

Proposition 1. The boundary equilibrium $E_1=(0,0)$ is an unstable equilibrium.

Proof. Letting $E_1=(0,0)$ into equation (8), we get the Jacobian matrix at E_1:

$$J(E_1)=\begin{pmatrix}(E-G) & 0 \\ 0 & (D-H)\end{pmatrix}.$$

By calculation we get two eigenvalues of $J(E_1):\lambda_1=E-G,\lambda_2=D-H$. If the conditions $E>G$ and $D>H$ are satisfied, $|\lambda_{1,2}|>0$ holds. So we can conclude the equilibrium E1 is unstable.

Proposition 2. The boundary equilibrium $E_2=(0,1)$ is an unstable equilibrium.

Proof. Taking $E_2=(0,1)$ into equation (8), we get the Jacobian matrix at E_2:

$$J(E_2)=\begin{pmatrix}(N+E-G) & 0 \\ 0 & -(D-H)\end{pmatrix}.$$

$J(E_2)$ has two eigenvalues: $\lambda_1=N+E-G$, $\lambda_2=-(D-H)$. From the conditions $E>G$ and $D>H$, it follows that $|\lambda_1|>0$ and $|\lambda_2|<0$. We conclude the equilibrium E_2 is a saddle point and unstable.

Proposition 3. The boundary equilibrium $E_3=(1,0)$ is an unstable equilibrium.

Proof. Letting $E_3=(1,0)$ into equation (8), we get the Jacobian matrix at E_3:

$$J(E_3)=\begin{pmatrix}-(E-G) & 0 \\ 0 & (M+D-H)\end{pmatrix}.$$

By calculation, we obtain two eigenvalues of $J(E_2):\lambda_1=-(E-G),\lambda_2=(M+D-H)$. From the conditions $M,N>0$, it follows that $|\lambda_1|<0$ and $|\lambda_2|>0$. So the equilibrium E_3 is a saddle point and unstable too.

Proposition 4. The boundary equilibrium $E_4=(1,1)$ is a stable equilibrium.

Proof. Letting $E_4=(1,1)$ into equation (8), we get the Jacobian matrix at E_4:

$$J(E_4)=\begin{pmatrix}-(N+E-G) & 0 \\ 0 & -(M+D-H)\end{pmatrix},$$

which has two eigenvalues: $\lambda_1=-(N+E-G)$, $\lambda_2=-(M+D-H)$. From the conditions $E>G$ and $D>H$, it follows that $|\lambda_1|<0$ and $|\lambda_2|<0$. So the equilibrium E_4 is stable.

To investigate the stability of $E_5=\left(\frac{D-H}{M},\frac{E-G}{N}\right)$, we denote $r=x-\frac{D-H}{M}$ and $z=y-\frac{E-G}{N}$. Then the system (5) is converted to the following model:

$$\begin{cases}\frac{dr}{dt}=\left(\frac{D-H}{M}-r\right)\left(\frac{S-F}{M}-r\right)z \\ \frac{dz}{dt}=\left(\frac{E-G}{N}-z\right)\left(\frac{R-C}{N}-z\right)r\end{cases} \quad (9)$$

Therefore we can study the zero point's stability in system (9), which equals to investigate the stability of E_5 in system (5). The one order linear approximation of system (9) is:

$$\begin{cases}\frac{dr}{dt}=\frac{(D-H)(S-F)}{M^2}z \\ \frac{dz}{dt}=\frac{(E-G)(R-C)}{N^2}r\end{cases} \quad (10)$$

Then we get the eigenvalue equation of system (10): $\lambda^2-K=0$, in which $K=\frac{(D-H)(S-F)(E-G)(R-C)}{M^2N^2}$. Hence, if $K>0$, zero solution in system (9) is asymptotic stability, which means E_5 is asymptotic stability too in system (5). If $K<0$, zero solution in system (9) is unstable, which means E_5 in system (5) is unstable too.

Proposition 5. If $K>0$, the interior equilibrium $E_5=\left(\frac{D-H}{M},\frac{E-G}{N}\right)$ is an asymptotic stable equilibrium. If $K<0$, E_5 is unstable in system (5).

3.2 Stability in system (6)

Taking $u=x-\frac{D-H}{M}$ and $v=y-\frac{E-G}{N}$ into system (6), we can obtain the following model:

$$\begin{cases}\frac{du}{dt}=\left(\frac{D-H}{M}-u\right)\left(\frac{S-F}{M}-u\right)v(t-\tau_1) \\ \frac{dv}{dt}=\left(\frac{E-G}{N}-v\right)\left(\frac{R-C}{N}-v\right)u(t-\tau_2)\end{cases} \quad (11)$$

And we get the one order linear approximation of system (11) is:

$$\begin{cases} \frac{du}{dt} = \frac{(D-H)(S-F)}{M^2} v(t-\tau_1) \\ \frac{dv}{dt} = \frac{(E-G)(R-C)}{N^2} u(t-\tau_2) \end{cases} \quad (12)$$

Then we work out the eigenvalue equation of system (12):

$$\lambda^2 - Ke^{-\lambda\tau} = 0, \quad (13)$$

which $K = \frac{(D-H)(S-F)(E-G)(R-C)}{M^2 N^2}$,

$\tau = \tau_1 + \tau_2$.

Proposition 6. If

$$\tau = \frac{1}{\sqrt{K}}\left[\cos^{-1}\left(-\frac{|K|}{K} + 2n\pi\right)\right] \quad (n = 0,1,2,\cdots),$$

equation (13) has a couple of pure imaginary roots $\lambda = \pm\sqrt{|K|}i$.

Proof. We assumed that $\lambda = i\varepsilon$ is one root of equation (13). Taking $\lambda = i\varepsilon$ into equation (13), we obtain $\varepsilon^2 + K(\cos(\varepsilon\tau) - i\sin(\varepsilon\tau)) = 0$. Then we separate the imaginary and real parts:

$$\begin{cases} \varepsilon^2 = -K\cos(\varepsilon\tau) \\ 0 = K\sin(\varepsilon\tau) \end{cases} \quad (14)$$

Then we get $\varepsilon^4 - K^2 = 0$, and obtain one positive solution $\varepsilon_0 = \sqrt{|K|}$. And taking it into equation (14), we obtain $\varepsilon_0{}^2 = -K\cos(\varepsilon_0\tau)$. Therefore,

$$\tau = \frac{1}{\varepsilon_0}\left[\cos^{-1}\left(-\frac{\varepsilon_0^2}{K}\right) + 2n\pi\right] = \frac{1}{\sqrt{K}}\left[\cos^{-1}\left(-\frac{|K|}{K} + 2n\pi\right)\right]$$

$(n = 0,1,2,\cdots)$

Proposition 7. The real part of the roots in equation (13): $\mathrm{Re}\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1} > 0$.

Proof. We differentiate the equation (13) on τ:

$$2\lambda\frac{d\lambda}{d\tau} + Ke^{-\lambda\tau}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right) = 0,$$

Then we work out:

$$\frac{d\lambda}{d\tau} = -\frac{\lambda Ke^{-\lambda\tau}}{2\lambda + K\tau e^{-\lambda\tau}}.$$

And get the reciprocal of $\frac{d\lambda}{d\tau}$:

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{2\lambda e^{\lambda\tau} + K\tau}{\lambda K}. \quad (15)$$

Replacing λ in equation (15) with $\lambda = \sqrt{|K|}i$, and making $\tau = \tau_0$, we obtain:

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\substack{\lambda=\sqrt{|K|}i \\ \tau=\tau_0}} = -\frac{2\cos(\sqrt{|K|}\tau_0)}{K} - \left(\frac{\sin(\sqrt{|K|}\tau_0)}{K} - \frac{\tau_0}{\sqrt{|K|}}\right)i$$

From the equation (14), the real part of $\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\substack{\lambda=\sqrt{|K|}i \\ \tau=\tau_0}}$ is:

$$-\frac{2\cos(\sqrt{|K|}\tau_0)}{K} = -\frac{2K\cos(\sqrt{|K|}\tau_0)}{K^2} > 0$$

So if $\tau \in (0,\tau_0)$, all solutions of equation (13) have negative real part. When $\tau > \tau_0$, the real part of more than one solution is positive.

Proposition 8. If $\tau \in (0,\tau_0)$, E_5 is local asymptotic stability in system (7). If $\tau \in (\tau_0,+\infty)$, E_5 is unstable. If $\tau = \tau_0$, Hopf bifurcation may appear at E_5 in system (7).

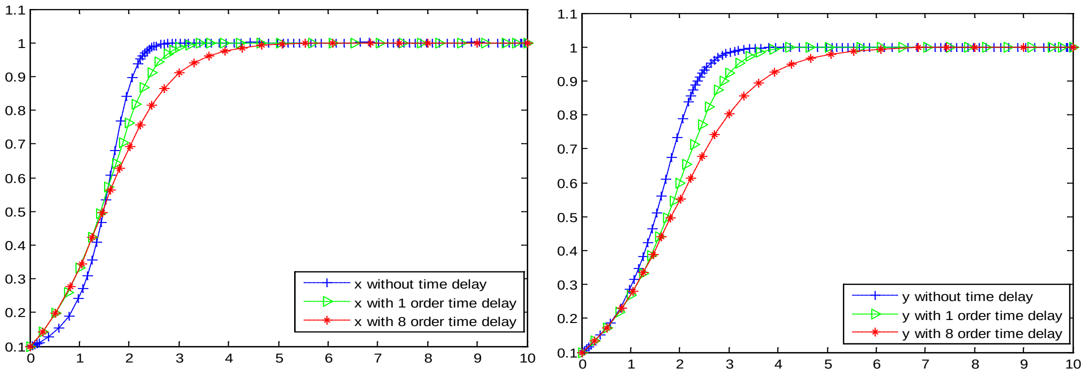
In the same way, we investigate the stability of other equilibrium points in system (7) and have following conclusions.

Proposition 9. The boundary equilibrium $E_1 = (0,0)$, $E_2 = (0,1)$ and $E_3 = (1,0)$ are unstable equilibrium points of system (7).

4. Numerical simulations

In this section, we make numerical simulations to show the influence of the time delay on the dynamic system. To observe the impact of time delay obviously, the numerical simulations by computer are used on both the system with time delay and the system without time delay. In our numerical experiments of simulation, some parameters are fixed: $S-F=3$, $E-G=0$, $D-H=1$. The span of time is from 0 to 10.

A) $x_0 = y_0 = 0.1$



B) $x_0 = y_0 = 0.01$

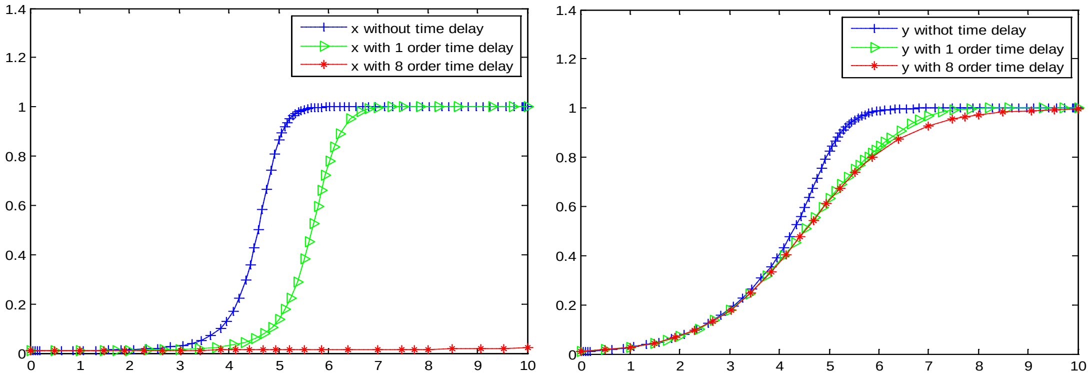
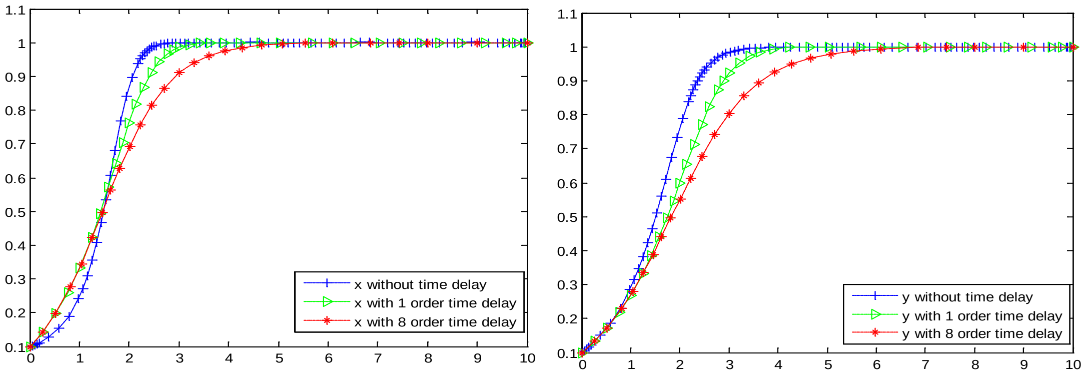


Figure 1. Sensitivity of the system to time delay

Figure1 is about the adjustments of the system (5) and (6) with respect to three different time delay. Figure 1 (A) shows the adjustment diagrams at $x_0 = y_0 = 0.1$: both x and y converges to 1, while the speed to the stable point is various with the time delay. Figure 1 (B) shows the adjustment diagrams at $x_0 = y_0 = 0.01$: y converges to the stable point with various speeds while the convergence of x is bifurcate. In figure 1 (A), system with the 8 order of time lag can get the stable point earlier than system with 1 order time delay while the system without time delay is the last one to arrive the stable point. So the order of time lag is larger, the speed of adjustment to stable state is quicker. So the observations of figure1 (A) and (B) tell system (6) without time delay reach the stable point later than system (7) with time lag.

A) $R - C = 6$



B) $R - C = 3$

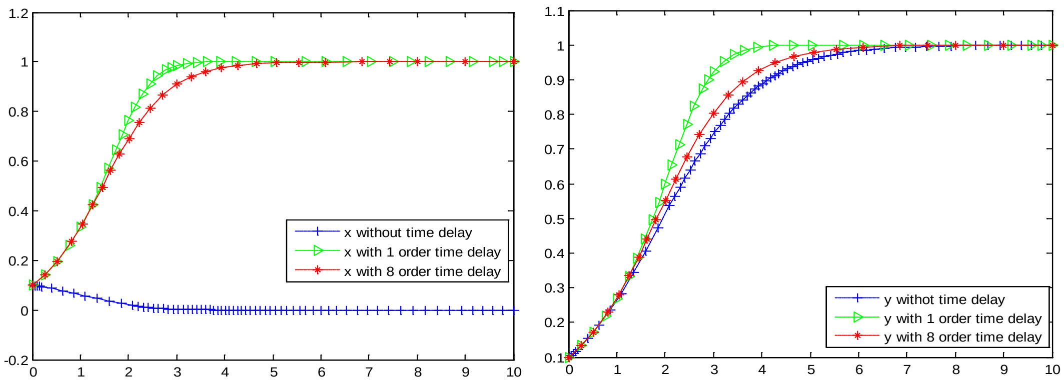
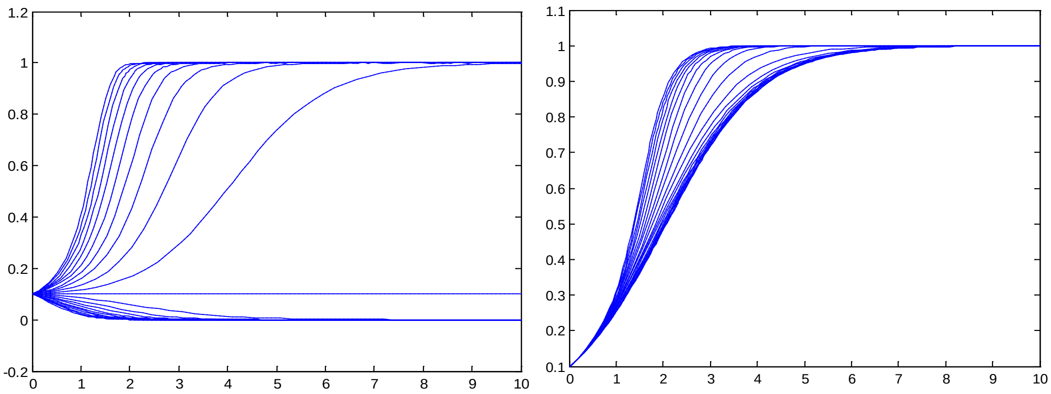


Figure 2. Sensitivity of the system to initial conditions

System (6) and (7) are sensitive to initial conditions. Figure 2 (A) shows the adjustment diagrams for the initial condition: $R-C=6$. Figure 2 (A) and Figure 1 (A) tell system (6) and (7) converge to the initial state $(1,1)$ with various speed. Figure 2 (B) shows the adjustment diagrams for the initial condition: $R-C=3$. In Figure 2 (B) and Figure 1 (B), x in the systems without time delay converge to the points $(0,0)$.

Figure 3 and Figure 4 further demonstrate the influence of initial conditions on the stability of equilibrium. Figure 3 displays the movement of system (6) on different initial conditions. Figure 3 (A) and (B) show the adjustment of system (6) if $(R-C)$ varies from -10 to 10. While Figure 3 (C) and (D) record portraits with different value of (x_0, y_0) if $(R-C)$ varies from -10 to 10. Fig.3 tells the variance of $(R-C)$ can lead system (6) convergence to different equilibrium point. If $(R-C)>0$, system (6) converges to point $(1,1)$. If $(R-C)<0$, system (6) converges to a point $(0,1)$. And the variance of initial point (x_0, y_0) may have impact on the speed of convergence. From Fig.3 (C) and (D) we see system (6) with $x_0=0.1$ and $y_0=0.1$ can reach to $(1,1)$ more quickly than one with $x_0=0.01$ and $y_0=0.01$. While it is on opposite side at the equilibrium point $(0,0)$.

A) Bifurcation diagrams of x B) Bifurcation diagrams of y



C) $x_0 = y_0 = 0.01$ D) $x_0 = y_0 = 0.1$

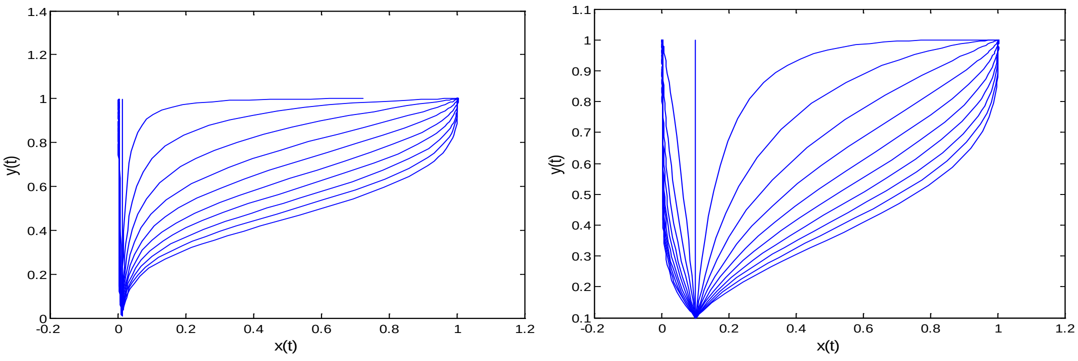
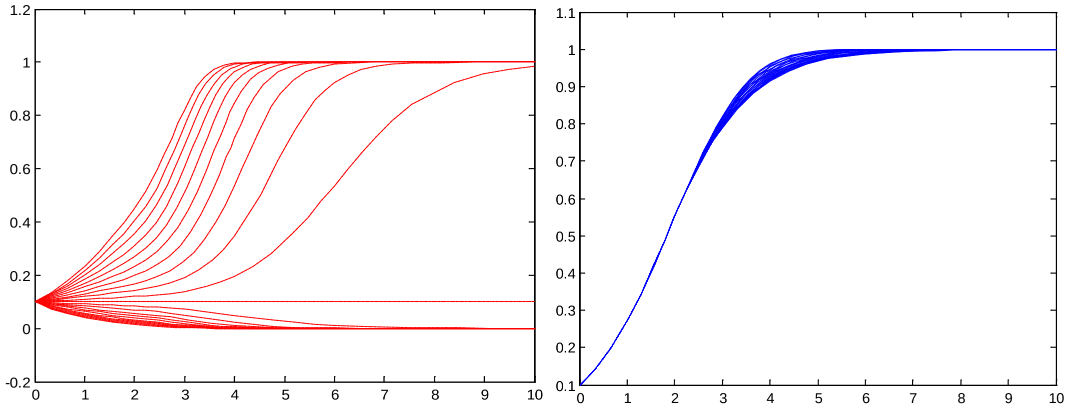


Figure 3. Stability of the system without time delay to initial conditions

Figure 4 displays the movement of system (7) on different initial conditions. Figure 4 (A) and (B) record the adjustment of system (7) if $(R-C)$ varies from -10 to 10. While Figure 4 (C) and (D) show portraits with various value of (x_0, y_0) if $(R-C)$ varies from -10 to 10. Figure 4 shows the variance of $(R-C)$ can lead system (7) convergence to different equilibrium point. If $(R-C)>0$, system (7) converges to point $(1,1)$. If $(R-C)<0$, system (7) converges to point $(0,1)$. And the variance of initial point (x_0, y_0) also produces effect on the speed of convergence in system (7). Observations of Figure 4 (C) and (D) tell system (7) with $x_0 = 0.01$ and $y_0 = 0.01$ converge to the equilibrium point more rapidly than one with $x_0 = 0.1$ and $y_0 = 0.1$.

A) Bifurcation diagrams of x B) Bifurcation diagrams of y



C) $x_0 = y_0 = 0.01$ D) $x_0 = y_0 = 0.1$

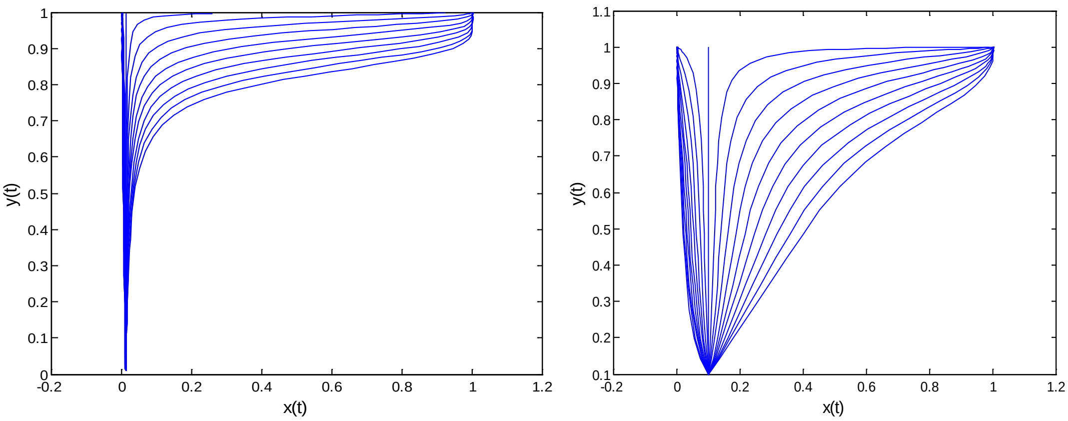


Figure 4. Stability of the system with time delay to initial conditions

5. Conclusion

In this paper, we consider two evolutionary game models: one is with time delay, the other is without time delay. To understand the effect of time delay on evolutionary game, we analyze the stability of the systems with and without time delay and make comparisons. We find that two systems have the same 5 equilibrium points and stability. Though time delay has no impact on the situation of the final stable state, it may change the speed of system convergence to the stable points. The numerical simulations of this paper verify that time delay and initial conditions have influence on the speed of convergence. So the conclusions of this paper mean that players will make more reasonable decisions if they make full use of the present and pass information in the evolutionary game.

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