Quantum Fluctuation in a Mesoscopic Inductor Coupled Circuit with Dissipation

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Abstract
This paper quantizes the mesoscopic inductor coupled circuit with dissipation by using the methods of normalized canonical transformation and unitary transformation, giving the accurate Hamiltonian. The fluctuation of the dissipative mesoscopic inductor coupled circuit is investigated. It shows that the average values for all the charges and currents are zero, but their mean-square values are not all zero when the circuit has no power. We also find that the zero point fluctuations of charges and currents exist in this system. It reveals that when $t \to \infty$, the fluctuations of the charges and currents would attenuate as time evolving.
Key words: MESOSCOPIC CIRCUIT WITH DISSIPATION, QUANTIZATION, QUANTUM FLUCTUATION

1. Introduction

Mesoscopic physics is developed rapidly with the abundant investigations of the action of carriers in solid, especially in the investigation of the action of electron in disordered system in 1970s and 1980s. Recently mesoscopic semiconductor, mesoscopic superconductor and mesoscopic magnetic are becoming more and more important in both basic physics and industrial application. It is well known that many problems in physic lie in many fields， such as quantum measurement, quantum computation, and quantum optic theory which is not only related to investigating the property of mesoscopic circuits [1,2,3,4], but also to the mesoscopic circuit involve several aspects[5]. And the research on quantum effects of electric circuits will be helpful to the miniaturization of integrate circuits and electric components. The Ref. [6] offered the quantum fluctuation at infinite temperature based on the thermodynamics theory, the Refs.[7] discussed the Coulomb blockade of tunneling, Bloch oscillations in mesoscopic circuit with charges quantized. The Ref.[8] discussed the dynamic process of mesoscopic LC circuit evolving from initial vacuum state to coherent state with external signal, considering the coupling energy aroused by the interference superposition of the wave functions of the electrons in the plates of mesoscopic capacitor. Mesoscopic circuit also plays an important role in quantum information. Experiments have already been reported using coupled persistent current qubits, where the mesoscopic circuit is a primary device [9, 10].

In fact, a quantum system loses its quantum characteristics such as nonlocality, entanglement, and coherence if it is open to the environment. Quantum dissipation is also investigated in various fields. Ref. [11] invests the quantum fluctuations of mesoscopic dissipative RLC circuit. But they only do that at thermal equilibrium without considering the influence of the initial state of the circuit and conclude that the quantum fluctuations of both charge and current approach to infinity at $t \to \infty$ which we think are not rational.

As coupling phenomenon is common in practical circuits, we investigate the mesoscopic inductor coupled dissipative circuit, considering the dissipative factor. The investigation of the mesoscopic inductor coupled circuits, the dissipation should be considered. In previous works, quantization based on that p and q are commutation, so when $t \to \infty$, and the fluctuation of current would be approaching to ∞, which disobey the principle of uncertainty. In this paper, we obtain the right quantization by canonical transformation [12], to eliminate the coupled item by unitary transformation, and investigate the quantum effects of the system by the right Hamiltonian of the circuit.

2. Quantization

The figure below shows the circuit:

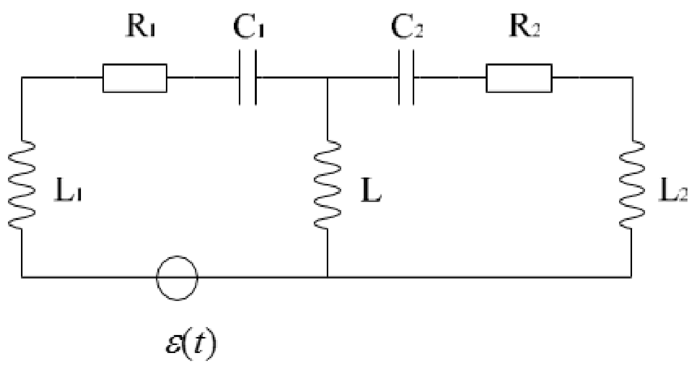

Figure 1. Inductor Coupled circuit with dissipation

According to Kirchoff's Law, the classic equations of motion can be defined as follow:

$$L_1\frac{d^2q_1}{dt^2}+R_1\frac{dq_1}{dt}+\frac{q_1}{C_1}+L_1\frac{d^2(q_1-q_2)}{dt^2}=\varepsilon(t) \tag{1}$$

$$L_2\frac{d^2q_2}{dt^2}+R_2\frac{dq_2}{dt}+\frac{q_2}{C_2}-L_1\frac{d^2(q_1-q_2)}{dt^2}=0 \tag{2}$$

Set $p_1=\frac{dq_1}{dt}, p_2=\frac{dq_2}{dt}$, from formulas (1) and (2)we can obtain:

$$\begin{aligned}\dot{p}_1=&\frac{\varepsilon(t)(L_2+L)}{L_1L_2+L(L_1+L_2)}-\frac{R_1p_1(L_2+L)}{L_1L_2+L(L_1+L_2)}-\frac{q_1(L_2+L)}{C_1[L_1L_2+L(L_1+L_2)]}\\&+\frac{R_2p_2L}{L_1L_2+L(L_1+L_2)}+\frac{q_2L}{C_2[L_1L_2+L(L_1+L_2)]}\end{aligned} \tag{3}$$

$$\begin{aligned}\dot{p}_2=&\frac{\varepsilon(t)L}{L_1L_2+L(L_1+L_2)}-\frac{R_2p_2(L_1+L)}{L_1L_2+L(L_1+L_2)}-\frac{q_2(L_2+L)}{C_2[L_1L_2+L(L_1+L_2)]}\\&-\frac{R_1p_1L}{L_1L_2+L(L_1+L_2)}-\frac{q_1L}{C_1[L_1L_2+L(L_1+L_2)]}\end{aligned} \tag{4}$$

Set

$$\frac{1}{M_1}=\frac{L_2+L}{L_1L_2+L\left(L_1+L_2\right)},\frac{1}{M_2}=\frac{L_1+L}{L_1L_2+L\left(L_1+L_2\right)}$$

$$\lambda_1=\frac{R_1}{M_1}\qquad \lambda_2=\frac{R_2}{M_2}$$

p and q are not commutation, and the following commutation relations should be satisfied. From formulas (3) and (4) we can obtain:

$$\frac{\partial\dot{q}_1}{\partial q_1}+\frac{\partial\dot{p}_1}{\partial p_1}=-\frac{R_1}{M_1} \tag{5}$$

$$\frac{\partial\dot{q}_2}{\partial q_2}+\frac{\partial\dot{p}_2}{\partial p_2}=-\frac{R_2}{M_2} \tag{6}$$

i.e.

$$\left[q_1,p_1\right]=\frac{ih}{M_1}\exp\left(-\lambda_1t\right) \tag{7}$$

$$[q_2, p_2] = \frac{ih}{M_2}\exp(-\lambda_2 t) \quad (8)$$

In a dissipative system, equation (5) and equation (6) imply that q and p cannot be contracted as common canonical variables, so the quantization conditions should be modified as follow

$$Q_1 = q_1 \exp(\frac{\lambda_1 t}{2}) \quad (9)$$

$$P_1 = M_1\dot{Q}_1 = (M_1 P_1 + \frac{R_1 q_1}{2})\exp(\frac{\lambda_1 t}{2}) \quad (10)$$

$$Q_1 = q_1 \exp(\frac{\lambda_1 t}{2}) \quad (11)$$

$$P_2 = M_2\dot{Q}_2 = (M_2 P_2 + \frac{R_2 q_2}{2})\exp(\frac{\lambda_1 t}{2}) \quad (12)$$

Equations (9)-(12) denote that P_j , Q_j are common canonical variables:

$$\frac{\partial \dot{Q}_j}{\partial Q_j} + \frac{\partial \dot{P}_j}{\partial P_j} = 0 \qquad (j = 1, 2)$$

$$[Q_1, P_1] = [Q_2, P_2] = ih \quad (13)$$

Q represents canonical charge, P represents canonical current. From the modification above, equation (1) and equation (2) can be rewritten as

$$L_1\ddot{Q}_1 + (\frac{1}{C_1} - \frac{R_1^2}{4M_1})Q_1 + L(\ddot{Q}_1 - \ddot{Q}_2) = \varepsilon(t)\exp(\frac{\lambda_1 t}{2}) \quad (14)$$

$$L_2\ddot{Q}_2 + (\frac{1}{C_2} - \frac{R_2^2}{4M_2})Q_2 - L(\ddot{Q}_1 - \ddot{Q}_2) = 0 \quad (15)$$

namely $\varepsilon(t) = 0$,according to Hamiltonian's canonical equation, the Hamiltonian of the system can be obtained as:

$$H = \frac{P_1^2}{2L_\alpha} + \frac{P_2^2}{2L_\beta} + \frac{1}{2}k_1 Q_1^2 + \frac{1}{2}k_2 Q_1^2 + \frac{L}{M_1 M_2}P_1 P_2 \quad (16)$$

Where

$$k_1 = \frac{1}{C_1} - \frac{R_1^2}{4M_1}, k_2 = \frac{1}{C_2} - \frac{R_2^2}{4M_2}$$

$$\frac{1}{L_\alpha} = \frac{1}{M_1} + \frac{L}{M_1^2}, \frac{1}{L_\beta} = \frac{1}{M_2} + \frac{L}{M_2^2}$$

Formula (16) describes two coupled quantum harmonic oscillator.

The following unitary operator to construct linear transformation is given to eliminate the coupled item.

$$S = \iint_{-\infty}^{+\infty} dq_1 dq_2 \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right| \quad (17)$$

Where $\left| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle = |q_1, q_2\rangle$ is coordinate eigenstate, and

$$a = \left(\frac{L_\beta}{L_\alpha}\right)^{1/4}\cos(\frac{\phi}{2}), b = \left(\frac{L_\beta}{L_\alpha}\right)^{1/4}\sin(\frac{\phi}{2})$$

$$c = -\left(\frac{L_\beta}{L_\alpha}\right)^{-1/4}\sin(\frac{\phi}{2}), d = \left(\frac{L_\beta}{L_\alpha}\right)^{-1/4}\cos(\frac{\phi}{2}) \quad (18)$$

$$\phi = \arctan\left[\frac{2L}{M_1 M_2(k_1 A_2^2 - k_2 A_2^{-2})}\right] \quad (19)$$

Following formulae can be easily proved

$$S^{-1}Q_1 S = aQ_1 + bQ_2$$
$$S^{-1}Q_2 S = cQ_1 + dQ_2 \quad (20)$$

$$S^{-1}P_1 S = dP_1 - cP_2$$
$$S^{-1}P_2 S = -bP_1 + aP_2 \quad (21)$$

Under S transformation, transformed Hamiltonian H' can be obtained:

$$H' = S^{-1}HS = \frac{Q_1^2}{2A_1} + \frac{Q_2^2}{2A_2} + \frac{1}{2}A_1\omega_1^2 P_1^2 + \frac{1}{2}A_2\omega_2^2 P_2^2 \quad (22)$$

$$\frac{1}{A_1} = k_1 a^2 + k_2 c^2, \frac{1}{A_2} = k_1 b^2 + k_2 d^2$$

$$\omega_1^2 = \frac{1}{A_1}(\frac{1}{L_\alpha}d^2 + \frac{1}{L_\beta}b^2 + \frac{2L}{M_1 M_2}db)$$

$$\omega_2^2 = \frac{1}{A_2}(\frac{1}{L_\alpha}c^2 + \frac{1}{L_\beta}a^2 + \frac{2L}{M_1 M_2}ca) \quad (23)$$

Introduce annihilation operator and creation operator as follow:

$$a_j = \sqrt{\frac{A_j\omega_j}{2h}}(Q_j + \frac{i}{A_j\omega_j}P_j) \qquad (j = 1, 2)$$

$$a_j^+ = \sqrt{\frac{A_j\omega_j}{2h}}(Q_j - \frac{i}{A_j\omega_j}P_j) \quad (24)$$

According to $[Q_1, P_1] = [Q_2, P_2] = ih$ operators satisfy the relation below:

$$\left[a_1, a_1^+\right] = \left[a_2, a_2^+\right] = 1$$

Then Equation（22）can be rewritten as:

$$H' = h\omega_1(a_1^+ a_1 + \frac{1}{2}) + h\omega_2(a_2^+ a_2 + \frac{1}{2}) \quad (25)$$

Equation (25) denotes that the coupled term has been eliminated by using transformation, and Hamiltonian evolves into the algebra sum of the Hamiltonians of two independent quantum linear harmonic oscillator. The spectrum and eigenvector of the mesoscopic inductor coupled circuit with dissipation can be written as follow

$$E_{n_1,n_2} = h\omega_1 (n_1 + \frac{1}{2}) + h\omega_2 (n_2 + \frac{1}{2}) \qquad (n_1, n_2 = 0,1,2) \tag{26}$$

$$\left| \psi_{n_1,n_2} \right\rangle = \left| n_1 \right\rangle \otimes \left| n_2 \right\rangle \qquad (n_1, n_2 = 0,1,2) \tag{27}$$

Where $|n_1\rangle$, $|n_2\rangle$ represent the eigenvectors of the system, and the frequency are ω_1 and ω_2 respectively. From the investigation above, the spectrum and eigenvector of this system are similar to that in capacitor coupled circuit[13]and inductor coupled lossless circuit [14]. The differences are the modulus (ω_1, ω_2) determined by the apparatus in this circuit and how they are connected.

In this part we calculate S . Using IWOP, the item can be written as:

$$S = \frac{2}{\sqrt{B}} \exp\left\{ \frac{1}{2B} \left(a^2 + b^2 - c^2 - d^2 \right) \left(a_1^{+2} - a_2^{+2} \right) + 4 \left(ac + bd \right) a_1^{+} a_2^{+} \right\}$$
$$:\exp\left\{ \left(a_1^{+} a_2^{+} \right) \left(G - I \right) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right\}: \tag{28}$$
$$\exp\left\{ \frac{1}{2B} \left(b^2 + d^2 - a^2 - c^2 \right) \left(a_1^{+2} - a_2^{+2} \right) - 4 \left(ab + cd \right) a_1^{+} a_2^{+} \right\}$$

Where

$$B = a^2 + b^2 + c^2 + d^2 + 2 = \left(\frac{L_\beta}{L_\alpha} \right)^{1/2} + \left(\frac{L_\beta}{L_\alpha} \right)^{-1/2} + 2$$

$$-\begin{pmatrix} a+d & b-c \\ c-d & a+d \end{pmatrix}$$

$$\overline{\left(\frac{L_\beta}{L_\alpha} \right)^{1/2} + \left(\frac{L_\beta}{L_\alpha} \right)^{1/2} +}$$

$$\times \begin{pmatrix} \left(\frac{L_\beta}{L_\alpha} \right)^{1/4} \cos\left(\frac{\phi}{2} \right) + \left(\frac{L_\beta}{L_\alpha} \right)^{-1/4} \cos\left(\frac{\phi}{2} \right) & \left(\frac{L_\beta}{L_\alpha} \right)^{1/4} \sin\left(\frac{\phi}{2} \right) + \left(\frac{L_\beta}{L_\alpha} \right)^{-1/4} \sin\left(\frac{\phi}{2} \right) \\ -\left(\frac{L_\beta}{L_\alpha} \right)^{1/4} \sin\left(\frac{\phi}{2} \right) - \left(\frac{L_\beta}{L_\alpha} \right)^{1/4} \sin\left(\frac{\phi}{2} \right) & \left(\frac{L_\beta}{L_\alpha} \right)^{1/4} \cos\left(\frac{\phi}{2} \right) \left(\frac{L_\beta}{L_\alpha} \right)^{1/4} \cos\left(\frac{\phi}{2} \right) \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{29}$$

From the investigation above, the system is a two-mode state. We suppose the initial state of the system is two-mode vacuum state $|00\rangle$, as the time that electric source acts on the circuit is very short, namely $t = \tau \to 0$, the state of the system evolves into two-mode squeezed vacuum state. The wave function can be written as follow:

$$\left| \psi \left(t = \tau \right) \right\rangle_{\tau \to 0} = S \left| 00 \right\rangle = \frac{2}{\sqrt{B}} \exp\left(\sigma_1 a_1^{+2} - \sigma_2 a_2^{+2} + \sigma_2 a_1^{+2} a_2^{+2} \right) \left| 00 \right\rangle \tag{30}$$

$$\sigma_1 = \frac{1}{B} \left(b^2 - c^2 \right), \sigma_2 = 4 \left(ac + bd - ab - cd \right) \tag{31}$$

Now, we investigate the quantum fluctuation with canonical transformation of the system in two-mode squeezed vacuum state. From Eqs. (20)-(24), we have

$$\langle Q_1\rangle = \langle Q_2\rangle = \langle P_1\rangle = \langle P_2\rangle = 0 \quad (32)$$

$$\langle (\Delta Q_1)^2\rangle = \langle 00|S^+Q_1^2S|00\rangle - \langle 00|S^+Q_1S|00\rangle = \frac{a^2h}{2A_1\omega_1} + \frac{b^2h}{2A_2\omega_2} \quad (33)$$

$$\langle (\Delta Q_2)^2\rangle = \langle 00|S^+Q_2^2S|00\rangle - \langle 00|S^+Q_2S|00\rangle = \frac{c^2h}{2A_1\omega_1} + \frac{d^2h}{2A_2\omega_2} \quad (34)$$

$$\langle (\Delta P_1)^2\rangle = \langle 00|S^+P_1^2S|00\rangle - \langle 00|S^+P_1S|00\rangle = \frac{d^2A_1\omega_1h}{2} + \frac{c^2A_2\omega_2h}{2} \quad (35)$$

$$\langle (\Delta P_2)^2\rangle = \langle 00|S^+P_2^2S|00\rangle - \langle 00|S^+P_2S|00\rangle = \frac{b^2A_1\omega_1h}{2} + \frac{a^2A_2\omega_2h}{2} \quad (36)$$

According to Eqs.(9)—(12), the fluctuations of currents and charges of initial variables are

$$\langle Q_1\rangle = \langle Q_2\rangle = \langle P_1\rangle = \langle P_2\rangle = 0 \quad (37)$$

$$\langle (\Delta q_1)^2\rangle = \frac{a^2h}{2A_1\omega_1} + \frac{b^2h}{2A_2\omega_2}\exp(-\lambda_1 t) \quad (38)$$

$$\langle (\Delta q_2)^2\rangle = \frac{c^2h}{2A_1\omega_1} + \frac{d^2h}{2A_2\omega_2}\exp(-\lambda_2 t) \quad (39)$$

$$\langle (\Delta p_1)^2\rangle = \frac{1}{M_1^2}\left(\frac{d^2A_1\omega_1h}{2} + \frac{c^2A_2\omega_2h}{2}\right)\exp(-\lambda_1 t) + \frac{R_1^2}{4M_1}\left(\frac{a^2h}{2A_1\omega_1} + \frac{b^2h}{2A_2\omega_2}\right)\exp(-\lambda_1 t) \quad (40)$$

$$\langle (\Delta p_2)^2\rangle = \frac{1}{M_2^2}\left(\frac{b^2A_1\omega_1h}{2} + \frac{a^2A_2\omega_2h}{2}\right)\exp(-\lambda_2 t) + \frac{R_2^2}{4M_2}\left(\frac{c^2h}{2A_1\omega_1} + \frac{d^2h}{2A_2\omega_2}\right)\exp(-\lambda_2 t) \quad (41)$$

Formulae above represent the fluctuations of charge and current of each loop in the circuit, and shows relation between the parameters of the apparatus of this system.

3. Conclusions

We obtain the Hamiltonian of the mesoscopic inductor coupled circuit with dissipation by means of canonical transformation and unitary transformation. The system is a two-mode state, equaling to the superposition of two harmonic oscillators. More than that, the expression of the spectrum and eigenvector of this system is similar to the mesoscopic capacitor coupled circuit with dissipation and the mesoscopic inductance coupled lossless circuit. Investigation of this paper indicates that if the initial state of the system is a two-mode vacuum state, the state will evolve into two-mode squeezed vacuum state while the time electric source affects on the system is very short, namely t → 0. We also investigate the quantum effects of the system in two-mode squeezed vacuum state, where formulae (33) to (41) denote that quantum fluctuation exists in each loop, and the fluctuation would be effeced by the condition of the circuit and the parameters (C_1, C_2, L_1, L_2, R_1, R_2, L) of each apparatus. The results indicates the fluctuations of one loop affected by the other loop. Eqs. (40) and (41) show that the dissipative apparatus affect the fluctuations of the loop in which they embed mainly. It also can obtain that the fluctuations of charge and current are attenuating as time involving.

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