Control Relevant Discretization of Nonlinear Delayed Non-Affine Systems Using the Matrix Exponential Algorithm

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Abstract

In this paper, a new discretization method to obtain the sampled data representation of the time delayed nonlinear non-affine continuous control system is proposed. This discretization method is based on the matrix exponential computation. The mathematical structure of the new discretization scheme is explored. Then it is applied to obtain the discrete form of the nonlinear non-affine time delayed continuous systems. The resulting time discretization method provides a finite dimensional representation for nonlinear control systems with time delayed non-affine input, thereby enabling the application of existing controller design techniques to such systems. The performance of the proposed discretization procedure is evaluated by means of the simulation study. In the simulation various sampling rates and time delay values are considered. The results demonstrate that the proposed discretization scheme can assure the system's accuracy requirements.

Key words: DISCRETIZATION OF NONLINEAR SYSTEM, TIME DELAY, NON-AFFINE INPUT, MATRIX EXPONENTIAL ALGORITHM.

1. Introduction

Nowadays, control relevant systems characterized by time delay problems are encountered in more and more situations. Efforts have been devoted to minimize the time delays. However, time delay cannot be eliminated totally due to its inherent nature, even with today's advanced technology. The reasons for this can be grouped into two major kinds. The first one is that time delays are becoming increasingly more widespread in control systems because of the convergence of communication and computational systems with traditional control engineering. Controller communication, especially communication over local area networks (LANs) or wide area networks (WANs), and complex computations resulting from digital controller implementations result in large time delays. In the case of WANs, such time delays are also time varying. As the communication and computational functions present in embedded control systems increase, the impact of time delays becomes more substantial and cannot be overlooked. The second reason is that control systems with non-negligible time delays exhibit complex behaviors because of their infinite dimensionality. Even a linear time-invariant (LTI) system with a constant time delay in the input or state has infinite dimensionality if expressed in the continuous time domain. As a result, controller design techniques developed over the last several decades for finite dimensional systems are difficult to apply to time delay systems with any degree of effectiveness. Control system design methods that explicitly account for the presence of time delays are required.

Effects of time delay on the stability and performance of control systems has drawn attention of many investigators in general process control systems, power systems, bilateral teleoperation systems, and networked control systems [1-5]. In general, time delay in active control systems causes unsynchronized application of the control forces, and this unsynchronization not only degrades the system performance, but also causes instability of the system response. Many of these models are also significantly nonlinear which motivates research in the control of nonlinear systems with time delay. A natural direction is to try to extend the ideas and results of nonlinear non-delay control to systems with delay.

Engineering studies dealing with time delay systems are extensive [6-9]. Cao and Wang investigated the problem of the robust stability for uncertain systems with time delay [10]. They used the Lyapunov method and quadratic stability theory to provide the delay-dependent stability criterion for the uncertain systems with time delay. Chemical reaction systems are often complex dynamic time-delay systems that have to operate successfully in the presence of uncertainties. Huang et al. presented a finite element collocation method to carry out flexibility analysis of chemical reaction systems with time delay [11]. The proposed method was combined with the linear quadratic regulator and Lagrange polynomial for the optimal solution of control variables and state variables respectively. Karimi and Gao presented a mixed H_2/H_∞ output-feedback control design methodology for second-order neutral linear systems with time varying state and input delays [12]. Delay-dependent sufficient conditions for the design of a desired control were given in terms of linear matrix inequalities. Moelja and Meinsma presented the H_2-optimal control problem of systems with multiple input delays [13]. Wu et al. investigated the problem of delay-dependent stability analysis for discrete-time Markovian jump neural networks with mixed time-delays [14].

Nowadays, modern nonlinear control strategies are usually implemented on a microcontroller or digital signal processor. As a direct consequence, the control algorithm has to work in discrete-time. For such digital control algorithms, one of the following time discretization approaches is typically used: time discretization of a continuous time control law designed on the basis of a continuous time system; and time discretization of a continuous time system resulting in a discrete time system and control law design in discrete time. It is apparent that the second approach is an attractive feature for dealing directly with the issue of sampling. It should be emphasized that in both design approaches time discretization of either the controller or the system model is necessary. Furthermore, notice that in the controller design for time delay systems, the first approach is troublesome because of the infinite-dimensional nature of the underlying system dynamics. As a result the second approach becomes more desirable and will be pursued in the present study.

In the field of the discretization, for the original continuous-time systems with time free case the traditional numerical techniques such as the Euler and Runge-Kutta methods have been used for getting the sampled-data representations [15]. But these methods need a small sampling time interval. Because it is necessary to meet the desired accuracy and they can not be applied to the large sampling period case. But due to the physical and technical limitation slow sampling is becoming inevitable. This paper presents a new matrix exponential approach to obtain the sampled data representation of the nonlinear non-affine delayed systems. The proposed discretization method can provide accurate and finite dimensional discretization results even in the case of big simpling period, so that the existing nonlinear controller design techniques can be applied to the nonlinear non-affine delayed systems.

2. Preliminary

In the present study the nonlinear continuous-time control systems with the time delayed non-affine input are considered with a state-space representation of the form:

$$\dot{x}(t) = f_0(x(t)) + g_1(x(t))u(t-D) + g_2(x(t))u(t-D)^2 + \ldots + g_m(x(t))u(t-D)^m$$
$$\text{or } \dot{x}(t) = f(x(t), u(t-D)) \tag{1}$$

where $x \in R^n$ is the vector of the states representing an open and connected set, $u \in R$ is the input variable, and D is the system's constant time-delay (deadtime) that directly affects the input. It is assumed that $f_0 : R^n \to R^n$, $g_i : R^n \to R^n$, $i = 1, 2, \cdots, m$ and $f \; R^n \times R \to R^n$ are smooth mappings.

An equidistant grid on the time axis with mesh $T = t_{k+1} - t_k > 0$ is considered where sampling interval is $[t_k, t_{k+1}) = [kT, (k+1)T)$ and T is the sampling period. Furthermore, we suppose the time-delay D and mesh T are related as follows.

$$D = qT + \gamma \tag{2}$$

where $q \in \{0,1,2,...\}$ and $0 \le \gamma < T$. Equivalently, the time-delay D is customarily represented as an integer multiple of the sampling period plus a fractional part of T.

It is also assumed that system (1) is driven by an input that is piecewise constant over the sampling interval, i.e. the zero-order hold assumption (ZOH) holds true.

For the ZOH, while $D = 0$,

$$u(t) = u(kT) \equiv u(k) = \text{constant} \tag{3}$$

for $kT \le t < kT + T$.

Under the ZOH assumption and the above notation, it is rather straightforward to verify that the "delayed" input variable attains the following values with expressions within the sampling interval, while $D \neq 0$,

$$u(t-D) = \begin{cases} u(kT - qT - T) \equiv u(k-q-1) & t \in [kT, kT+\gamma) \\ u(kT - qT) \equiv u(k-q) & t \in [kT+\gamma, kT+T) \end{cases} \tag{4}$$

3. Discretization of nonlinear non-affine delayed systems using matrix exponential algorithm

3.1. Discretization of nonlinear systems with delay-free input

Delay-free $D = 0$ nonlinear control systems are considered with state-space representations of the form:

$$\frac{dx(t)}{dt} = f(x(t)) + u(t)g(x(t))$$
$$x(0) = x_0 \tag{5}$$

Consider the sampling time interval $[t_k, t_{k+1})$, and suppose that $u(t) = u_k$, $t \in [t_k, t_{k+1})$. Let us denote,

$$\xi(t) = x(t) - x_k \tag{6}$$

where $x_k = x(t_k)$, $t \in [t_k, t_{k+1})$.

We then get the following approximation

$$f(x(t)) \approx f(x_k) + \frac{\partial f(x_k)}{\partial x}\xi(t) \tag{7}$$

$$g(x(t)) \approx g(x_k) + \frac{\partial g(x_k)}{\partial x}\xi(t) \tag{8}$$

Based on the above, within the time interval $[t_k, t_{k+1})$, Eq. (5) can be approximated through the following expression:

$$\begin{aligned}\dot{x}(t) = \dot{\xi}(t) &= f(x_k) + \frac{\partial f(x_k)}{\partial x}\xi(t) + (g(x_k) + \frac{\partial g(x_k)}{\partial x}\xi(t))u_k \\ &= (f(x_k) + g(x_k)u_k) + (\frac{\partial f(x_k)}{\partial x} + \frac{\partial g(x_k)}{\partial x}u_k)\xi(t) = \tilde{f}_k + J_k\xi(t)\end{aligned} \tag{9}$$

where

$$\tilde{f}_k = \tilde{f}(x_k, u_k) = f(x_k) + g(x_k)u_k \tag{10}$$

$$J_k = J(x_k, u_k) = \frac{\partial f(x_k)}{\partial x} + \frac{\partial g(x_k)}{\partial x}u_k \tag{11}$$

Rewriting Eq. (9), we obtain

$$\dot{x}(t) = \dot{\zeta}(t) = \tilde{f}_k + J_k\xi(t), \quad \xi(t_k) = 0 \tag{12}$$

Assume that $N > 0$ is an integer number and denote that

$$h = \frac{t_{k+1} - t_k}{N} \tag{13}$$

Using the new step of discretization h, Eq. (12) can be replaced by the following equation:

$$\frac{\xi(t_k + (i+1)h) - \xi(t_k + ih)}{h} = \tilde{f}_k + J_k\xi(t_k + ih) \tag{14}$$

where $i \in (0, 1, ..., N-1)$. Then we can get

$$\xi(t_k + (i+1)h) = (\mathbf{I} + J_k h)\xi(t_k + ih) + h\tilde{f}_k,\ \xi(t_k) = 0 \quad (15)$$

where $\mathbf{I}$ is an identity matrix with the appropriate dimension. From Eq. (15) we can get:

$$\xi(t_{k+1}) = \xi(t_k + Nh) = h\sum_{i=0}^{N-1}(\mathbf{I} + J_k h)^i \tilde{f}_k \quad (16)$$

Consequently, the discretization form of Eq. (5) can be calculated using Eq. (17).

$$x_{k+1} = x_k + \xi(t_{k+1}) = x_k + h\sum_{i=0}^{N-1}(\mathbf{I} + J(x_k, u_k)h)^i \tilde{f}(x_k, u_k) \quad (17)$$

where functions $J(x_k, u_k)$ and $\tilde{f}(x_k, u_k)$ are defined in Eqs. (10-11).

The systems (5) can also be solved in another way that is different with the method of (17).

Here, introduce an extended vector,

$$\eta(t) = \begin{pmatrix} \xi(t) \\ 1 \end{pmatrix} \quad (18)$$

Then, system (12) can be rewritten in the form

$$\dot{\eta}(t) = C_k \eta(t),\ \eta(\mathrm{t_k}) = \begin{pmatrix} \bar{0} \\ 1 \end{pmatrix} = \eta_0 \quad (19)$$

where $C_k = \begin{pmatrix} J_k & \tilde{f}_k \\ \bar{0}^T & 0 \end{pmatrix} \in R^{(n+1)\times(n+1)}$, $\bar{0}$ is n dimensional zero column vector, and $\bar{0}^T$ is n dimensional zero row vector.

Solving Eq. (19), we can get that,

$$\eta(t_{k+1}) = e^{C_k \cdot (t_{k+1} - t_k)}\eta_0 \quad (20)$$

In [16] a computation method of the exponential of a matrix was presented. It will be reviewed briefly in the following.

Let Z be a square matrix and I the corresponding identity matrix. The exact formula

$$e^Z = \lim_{N\to\infty}\left(I + \frac{Z}{N}\right)^N \quad (21)$$

provides the truncated approximation

$$e^Z \cong \left(I + \frac{Z}{N}\right)^N \quad (22)$$

for a suitable value of N.

An improved form is

$$e^Z \cong \left(I + \frac{Z}{2^b}\right)^{2^b} \quad (23)$$

Combined (22) and (20), there is

$$e^{C_k \cdot (t_{k+1} - t_k)} \approx \left(I + \frac{C_k(t_{k+1} - t_k)}{N}\right)^N = \left(I + C_k h\right)^N \quad (24)$$

From (24) and (20), we can obtain

$$\eta(t_{k+1}) = \left(I + C_k h\right)^N \eta_0 \Rightarrow \eta(t_{k+1}) = \begin{pmatrix} I + hJ_k & h\tilde{f}_k \\ \bar{0}^T & 1 \end{pmatrix}^N \begin{pmatrix} \bar{0} \\ 1 \end{pmatrix} \Rightarrow \xi(t_{k+1}) = \begin{pmatrix} I & \bar{0} \end{pmatrix}\begin{pmatrix} I + hJ_k & h\tilde{f}_k \\ \bar{0}^T & 1 \end{pmatrix}^N \begin{pmatrix} \bar{0} \\ 1 \end{pmatrix} \quad (25)$$

where $\begin{pmatrix} I & \bar{0} \end{pmatrix} \in R^{n\times(n+1)}$.

Hence

$$x_{k+1} = x_k + \begin{pmatrix} I & \bar{0} \end{pmatrix}\begin{pmatrix} I + hJ_k & h\tilde{f}_k \\ \bar{0}^T & 1 \end{pmatrix}^N \begin{pmatrix} \bar{0} \\ 1 \end{pmatrix} \quad (26)$$

Eq. (26) can be used to obtain the discrete time form of nonlinear systems.

3.2. Discretization of nonlinear non-affine delayed systems

The discretization method presented in Section 3.1 can be extended to the case of nonlinear systems with delayed nonaffine input. Based on the preliminaries presented in Section 2, since the time delay is introduced, we should consider each sampling time interval $[kT, (k+1)T)$ as two subintervals, $[kT, kT+\gamma)$ and $[kT+\gamma, kT+T)$.

In order to apply the discretization method of (26), we should choose N_1 and N_2 firstly, and makes it meet the requirement

$$\frac{N_1}{N_2} \approx \frac{\gamma}{T-\gamma} \quad (27)$$

where N_1 and N_2 are both positive natural numbers. Eq. (27) indicates that the calculation step lengths of $[kT, kT+\gamma)$ and $[kT+\gamma, kT+T)$ are nearly identical.

Applying (26) for the subinterval $[kT, kT+\gamma)$, we get

$$x_{kT+\gamma} = x_{kT} + \begin{pmatrix} I & \bar{0} \end{pmatrix}\begin{pmatrix} I + h_1 J(x_{kT}, u_{kT-qT-T}) & h_1\tilde{f}(x_{kT}, u_{kT-qT-T}) \\ \bar{0}^T & 1 \end{pmatrix}^N \begin{pmatrix} \bar{0} \\ 1 \end{pmatrix} \quad (28)$$

where

$$h_1 = \frac{\gamma}{N_1} \quad (29)$$

$$\tilde{f}(x_{kT}, u_{kT-qT-T}) = f(x_{kT}, u_{kT-qT-T}) \quad (30)$$

$$J(x_{kT}, u_{kT-qT-T}) = \frac{\partial f(x_k, u_{kT-qT-T})}{\partial x} \quad (31)$$

Applying (26) for the subinterval $[kT+\gamma, kT+T)$, we get

$$x_{kT+T} = x_{kT+\gamma} + \begin{pmatrix} I & \bar{0} \end{pmatrix} \begin{pmatrix} I + h_2 J(x_{kT+\gamma}, u_{kT-qT}) & h_1 \tilde{f}(x_{kT+\gamma}, u_{kT-qT}) \\ \bar{0}^T & 1 \end{pmatrix}^N \begin{pmatrix} \bar{0} \\ 1 \end{pmatrix} \quad (32)$$

where

$$h_2 = \frac{T-\gamma}{N_2} \quad (33)$$

$$\tilde{f}(x_{kT+\gamma}, u_{kT-qT}) = f(x_{kT+\gamma}, u_{kT-qT}) \quad (34)$$

$$J(x_{kT+\gamma}, u_{kT-qT}) = \frac{\partial f(x_{k+\gamma}, u_{kT-qT})}{\partial x} \quad (35)$$

By using Eq. (28) and Eq. (32), the sampled data representation of the nonlinear non-affine delayed system can be obtained.

4. Simulation

The performance of the proposed time discretization method was evaluated by applying it to a nonaffine time delayed nonlinear system. Reference solutions for the system are required to validate the proposed time-discretization method. In this paper the Matlab ODE solver is used to obtain reference solutions. The discrete values obtained at every time step using the proposed time-discretization method are compared to the values obtained using the Matlab ODE solver at the corresponding time steps. The proposed discretization method is realized using Maple.

The system considered in this paper is assumed to be a nonlinear control system.

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_1 x_2 u^2 + 3\cos(x_2)x_1 \\ \dot{x}_2 &= x_2^2 u \end{aligned} \quad (36)$$

In this simulation the input u was assumed to be $u = \sin(1.5t)$. And we choose the parameters as $x_1(0) = 1.0$, $x_2(0) = -1.0$, $T = 0.01s$, $D = 0.006s$; and $x_1(0) = 1.0$, $x_2(0) = -1.0$, $T = 0.01s$, $D = 0.084s$, respectively. The results obtained by the Matlab ODE solver and the proposed discretization method of these two cases are shown in Table 1 and Table 2. Figs. 1 and 2 show the errors of the state x_1 and x_2 of these two cases, respectively. From the results it can be seen that the proposed discretization method can provide good enough results for nonlinear nonaffine time-delayed systems.

Table1. Discretization results of case 1

T=0.01s, D=0.006s, x1=1.0, x2=-1.0				
Steps	Matlab (x1)	Maple (x1)	Matlab (x2)	Maple (x2)
100	0.6907	0.6891	-0.6197	-0.6216
200	0.6117	0.6119	-0.4300	-0.4300
300	0.5661	0.5652	-0.5516	-0.5499
400	0.5563	0.5572	-0.9727	-0.9701
500	0.4937	0.4930	-0.6997	-0.7010
600	0.5576	0.5588	-0.4405	-0.4403
700	0.4997	0.4986	-0.5031	-0.5012
800	0.5573	0.5586	-0.9071	-0.8990
900	0.4691	0.4689	-0.7920	-0.7914
1000	0.5631	0.5637	-0.4616	-0.4611

Table2. Discretization results of case 2

T=0.01s, D=0.084s, x1=1.0, x2=-1.0				
Steps	Matlab (x1)	Maple (x1)	Matlab (x2)	Maple (x2)
100	0.6712	0.6698	-0.6508	-0.6529
200	0.6234	0.6228	-0.4330	-0.4331
300	0.5534	0.5525	-0.5299	-0.5282
400	0.5668	0.5675	-0.9484	-0.9450
500	0.4866	0.4860	-0.7365	-0.7376
600	0.5646	0.5657	-0.4478	-0.4477
700	0.4922	0.4911	-0.4869	-0.4853
800	0.5643	0.5658	-0.8719	-0.8629
900	0.4676	0.4676	-0.8311	-0.8301
1000	0.5635	0.5639	-0.4736	-0.4731

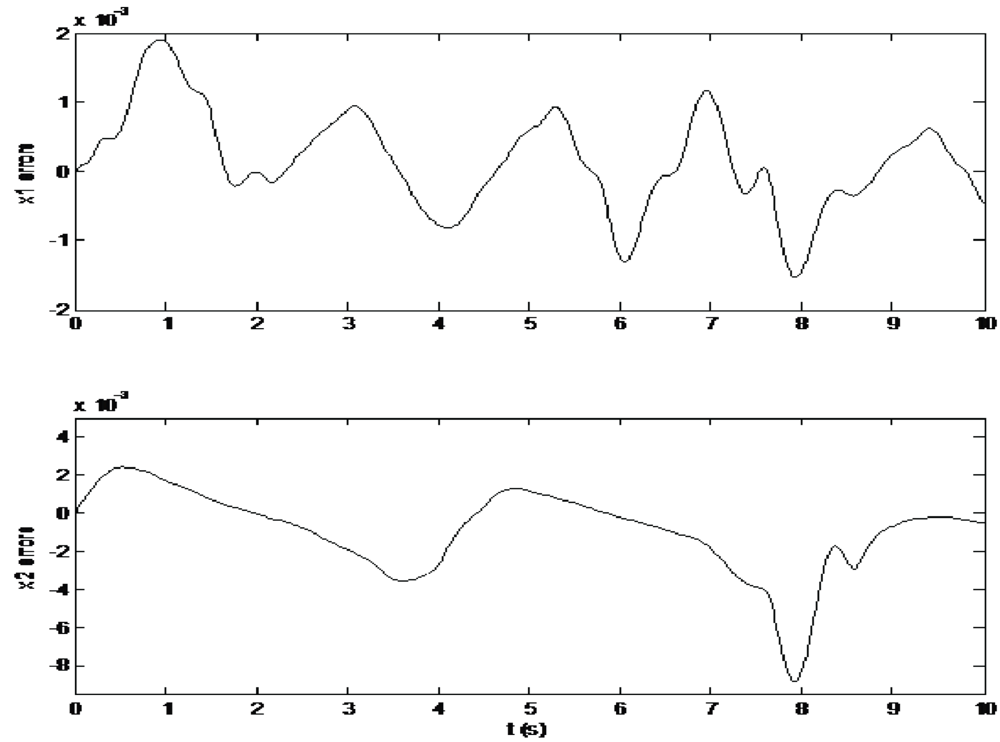

Figure 1. State error responses of the system in case 1

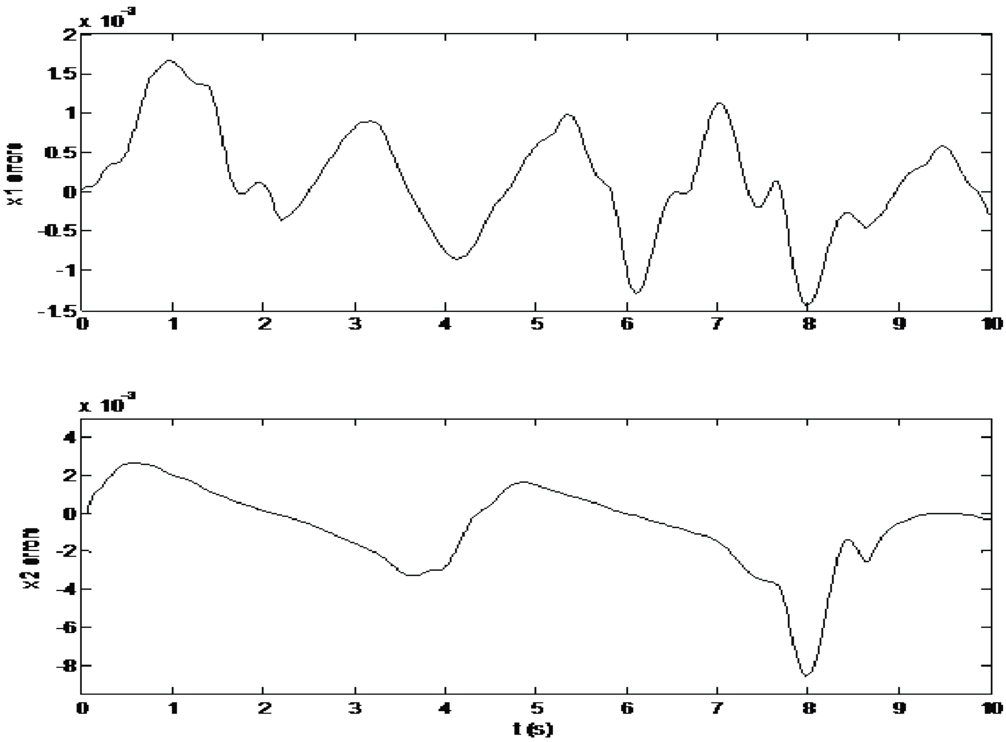

Figure 2. State error responses of the system in case 2

5. Conclusion

This paper presented an approach to obtain discrete-time representations of nonlinear control systems with nonaffine time-delay inputs in their control schemes. This proposed discretization algorithm is based on matrix exponential algorithm. The proposed scheme provided a finite-dimensional representation for nonlinear systems with nonaffine time-delay inputs enabling existing controller design techniques to be applied. The performance of the proposed discretization scheme was evaluated using a nonlinear system. These simulations demonstrate the accuracy of the proposed discretization method. Extension of the proposed approach to nonlinear systems with time-varying delay is feasible, and it will be the subject of future research.

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