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## **Determination of integral characteristics of stress state of the point during plastic deformation in conditions of volume loading**

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Abstract

It was set and solved the closed problem of the theory of plasticity in terms of volume loading. Circularity of solution does not complicate the task, but simplifies as six equations of kinematic part of a task gives to the solution at a choice of combinations of flat functions. According to the equations of connection, such requirements of deformations are imposed to the static part of a task at determination of stress.

Various solutions for the making components of a tensor of stress were obtained. Dependencies for integral characteristics of stress state of deformation point were shown. The kernel of the solution of intensity of normal stress, which characterizes the generalized point stress parameters was defined.

Key words: INTEGRATED CHARACTERISTICS, PLASTIC MEDIUM, STRESS STATE, INTENSITY, VOLUME LOADING

Introduction

At the solution of practical and a number of theoretical tasks there is a need to define the integrated characteristics of tension of a point. Among these are

the intensities of tangential normal stress  $T_i, \sigma_i$ . At the known values of components of stress tensor, intensities are defined by the following expressions

$$\sigma_i = \frac{1}{\sqrt{2}} \sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}, \tag{1}$$

$$T_i = \frac{1}{\sqrt{6}} \sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)},$$

Further the expressions (1) can be used for definition of mechanical characteristics of concrete steel grade in conditions of hot or cold processing in each point of the deformation zone.

Solution of space problem in analytical form causes great mathematical troubles. That is why there always appear questions connected with simplifications in statement and its solution [1]. Closed task of the theory of plasticity connected with definition of fields of tension and deformations, their correlation between each other, satisfaction of boundary conditions, complicates considerably the solution. It results in insuperable difficulties in the solution. However, it is possible to stop on the options when closed problem definition does not complicate, but simplifies a task. The system of the differential equations of ki-

nematic part of a task can be simplified if to use a combination of flat functions, and then through the equations of connection to come to static part when determining a tension of a point. At such scheme of solution, it is possible to consider the option satisfying all system of the equations of the theory of plasticity in the closed view. Thus, the static part of a task is proved and supported by the kinematic. Besides, at the solution of practical tasks it is not always possible to find reliable result for analytical determination of tension and deformations in transition zones from one part of deformation zone into the adjacent.

The figure 1 shows stress tensor components and contact surface of deformation zone in plan taking into account transition areas from one zone of plastic flow into another.

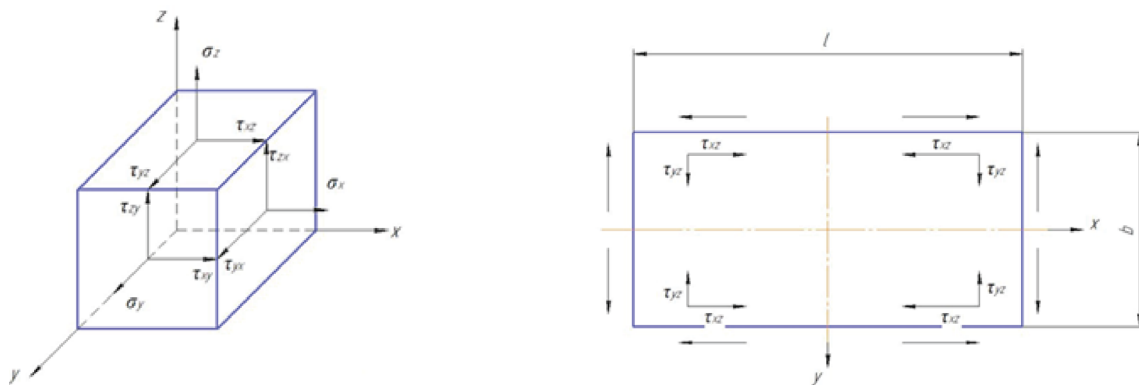


Figure 1. Stress tensor components and mechanism of tangential stress on the contact

At the transition areas tangential stresses  $\tau_{xz}$  and  $\tau_{yz}$  change along the value and direction. On the boundary line of metal flow, axes  $Y, Y, \tau_y$ , tangential stresses  $\tau_{xy}, \tau_{yz}, \tau_{zx}$  are equal to zero [2]. This allows to suppose that along these axes there is plane-strain condition.

### Problem statement

Taking into account the last notes, there considered the following statement of closed volume task of plastic theory including the equations [3]...[7]

#### 1. Equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (1)$$

#### 2. Generalized equilibrium equations

$$\frac{\partial^2 \tau_{xy}}{\partial x^2} - \frac{\partial^2 \tau_{xy}}{\partial y^2} = \frac{\partial^2}{\partial x \partial y} (\sigma_x - \sigma_y), \quad \frac{\partial^2 \tau_{yz}}{\partial y^2} - \frac{\partial^2 \tau_{yz}}{\partial z^2} = \frac{\partial^2}{\partial y \partial z} (\sigma_y - \sigma_z), \quad (2)$$

$$\frac{\partial^2 \tau_{xz}}{\partial x^2} - \frac{\partial^2 \tau_{xz}}{\partial z^2} = \frac{\partial^2}{\partial x \partial z} (\sigma_z - \sigma_x).$$

#### 3. Equations of connection

$$\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{\xi_x - \xi_y}{\gamma_{xy}} = F_1, \quad \frac{\sigma_y - \sigma_z}{2\tau_{yz}} = \frac{\xi_y - \xi_z}{\gamma_{yz}} = F_2, \quad \frac{\sigma_z - \sigma_x}{2\tau_{zx}} = \frac{\xi_z - \xi_x}{\gamma_{zx}} = F_3 \quad (3)$$

#### 4. Compatibility equations of strain rate

$$\frac{\partial^2 \xi_x}{\partial y^2} + \frac{\partial^2 \xi_y}{\partial x^2} = \frac{\partial^2}{\partial y \partial x} \gamma_{xy}, \quad \frac{\partial^2 \xi_y}{\partial z^2} + \frac{\partial^2 \xi_z}{\partial y^2} = \frac{\partial^2}{\partial z \partial y} \gamma_{yz}, \quad \frac{\partial^2 \xi_z}{\partial x^2} + \frac{\partial^2 \xi_x}{\partial z^2} = \frac{\partial^2}{\partial x \partial z} \gamma_{zx} \quad (4a)$$

$$\frac{\partial}{\partial z} \left( \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right) = 2 \cdot \frac{\partial^2 \xi_z}{\partial x \partial y}, \quad \frac{\partial}{\partial x} \left( \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x} \right) = 2 \cdot \frac{\partial^2 \xi_x}{\partial y \partial z},$$

$$\frac{\partial}{\partial y} \left( \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} \right) = 2 \cdot \frac{\partial^2 \xi_y}{\partial z \partial x} \quad (4b)$$

#### 5. Boundary conditions

$$\tau_{n1} = k_1 \cdot \text{Sin}(A_1 F_1 - 2\alpha_1), \quad \tau_{n2} = k_2 \cdot \text{Sin}(A_2 F_2 - 2\alpha_2), \quad (5)$$

$$\gamma_{n2} = 2 \cdot \beta_2 \cdot \text{Sin}(B_2 F_2 - 2\alpha_2), \quad \gamma_{n2} = 2 \cdot \beta_2 \cdot \text{Sin}(B_2 F_2 - 2\alpha_2).$$

In result of arrangement we have 19 unknown variables  $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}, \xi_x, \xi_y, \xi_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}, k_1, k_2, k_3, \beta_1, \beta_2, \beta_3, T_i)$  and 19 equations including 4 equations for boundary conditions (1)...(5), where  $k_1, k_2, k_3$  - are the resistance to flow shear in plains  $XY, YZ, ZX$ ;  $\beta_1, \beta_2, \beta_3$  - are the intensities of shear speed of deformation in the same plains. The task is statically determinable.

### Structure of solution

Use of flat functions or combinations of flat functions in the solution is defined by objective to simplify a space problem and fully satisfy the system of equations of plasticity theory (1)...(5). The generalized equations of balance (2), at known differences of normal tension, give mathematical possibility to characterize transition areas between zones of plastic current. The analysis shows that there are options, at which their application comes true at the solution of

space tasks of plasticity theory.

In the works [4]...[7] analytical solution of plane task of plasticity theory with the usage of harmonic functions is suggested. In the mentioned publications, it is shown that the transitions areas of contiguous zones are well described by generalized equilibrium equation (2). These equations are known in

literature [8]...[11] and allows to obtain real fields of tangential stress in the areas, where stresses change their operator.

After transformations from the equilibrium equation one may obtain the system connecting tangential stress with the remainder of normal ones [4], we will have

$$\begin{aligned} \frac{\partial^2 \tau_{xy}}{\partial x^2} - \frac{\partial^2 \tau_{xy}}{\partial y^2} + \frac{\partial^2 \tau_{yz}}{\partial z \partial x} - \frac{\partial^2 \tau_{xz}}{\partial z \partial y} &= \frac{\partial^2}{\partial x \partial y} (\sigma_x - \sigma_y), \\ \frac{\partial^2 \tau_{zy}}{\partial y^2} - \frac{\partial^2 \tau_{zy}}{\partial z^2} + \frac{\partial^2 \tau_{zx}}{\partial x \partial y} - \frac{\partial^2 \tau_{yx}}{\partial x \partial z} &= \frac{\partial^2}{\partial y \partial z} (\sigma_y - \sigma_z), \\ \frac{\partial^2 \tau_{xz}}{\partial z^2} - \frac{\partial^2 \tau_{zx}}{\partial x^2} + \frac{\partial^2 \tau_{yx}}{\partial z \partial x} - \frac{\partial^2 \tau_{yz}}{\partial z \partial y} &= \frac{\partial^2}{\partial z \partial x} (\sigma_z - \sigma_x). \end{aligned} \quad (6)$$

Expression (6) may be used if there exist the possibility to connect normal and residual stresses. To simplify (6) let us use the limitations, which are

determined by kinematic problem specification. In this case, mixed derivatives from tangential stresses according to the coordinates are equal to zero

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

hence

$$\tau_{yz} = f(x, y) + f(y, z), \quad \tau_{xz} = f(x, y) + f(z, x),$$

$$\tau_{zx} = f(z, y) + f(z, x), \quad \tau_{yx} = f(z, y) + f(x, y),$$

$$\tau_{xy} = f(z, x) + f(x, y), \quad \tau_{zy} = f(x, z) + f(y, z).$$

To satisfy boundary conditions in the stresses and acceptance of possible simplifications, we will have

$$\tau_{xy} = f(x, y), \quad \tau_{yz} = f(y, z), \quad \tau_{zx} = f(z, x). \quad (7)$$

The system

$$\begin{aligned} \frac{\partial^2 \tau_{xy}}{\partial x^2} - \frac{\partial^2 \tau_{xy}}{\partial y^2} &= \frac{\partial}{\partial x \partial y} (\sigma_x - \sigma_y) & \frac{\partial^2 \tau_{zy}}{\partial y^2} - \frac{\partial^2 \tau_{zy}}{\partial z^2} &= \frac{\partial^2}{\partial y \partial z} (\sigma_y - \sigma_z) \\ \frac{\partial^2 \tau_{xz}}{\partial z^2} - \frac{\partial^2 \tau_{zx}}{\partial x^2} &= \frac{\partial^2}{\partial z \partial x} (\sigma_z - \sigma_x) \end{aligned} \quad (8)$$

The system (8) is represented as the determining one in the system of equations of plasticity theory.

Connection of normal tangential stresses may be defined using equations of connection (3)

$$\sigma_x - \sigma_y = 2\tau_{xy} \cdot F_1, \quad \sigma_y - \sigma_z = 2\tau_{yz} \cdot F_2, \quad \sigma_z - \sigma_x = 2\tau_{zx} \cdot F_2. \quad (9)$$

During problem statement (1)...(5), we will have

$$\frac{\partial^2 \tau_{xy}}{\partial x^2} - \frac{\partial^2 \tau_{xy}}{\partial y^2} = \frac{\partial}{\partial x \partial y} 2 \cdot \tau_{xy} F_1, \quad \frac{\partial^2 \tau_{yz}}{\partial y^2} - \frac{\partial^2 \tau_{yz}}{\partial z^2} = \frac{\partial^2}{\partial y \partial z} 2 \cdot \tau_{yz} F_2$$

$$\frac{\partial^2 \tau_{zx}}{\partial z^2} - \frac{\partial^2 \tau_{zx}}{\partial x^2} = \frac{\partial^2}{\partial z \partial x} 2 \cdot \tau_{zx} F_3 \quad (10)$$

Such schemes simplify the solution, as in the equation there used one decision function  $\tau$ . Considering in (10) the products with goniometric functions [4]...[7], one may accept

$$\tau_{xy} \cdot F_1 = k_1 \cdot \text{Cos}(A_1 F_1), \tau_{yz} \cdot F_2 = k_2 \cdot \text{Cos}(A_2 F_2),$$

$$\tau_{zx} \cdot F_3 = k_3 \cdot \text{Cos}(A_3 F_3), \quad (11)$$

herein

$$F_1 = \text{ctg} A_1 F_1, \quad F_2 = \text{ctg} A_2 F_2, \quad F_3 = \text{ctg} A_3 F_3.$$

However real connection of normal and tangential stresses for space problem do not correspond to the correlations (9)...(11). This connection is more complicated, which may be seen during elementary analysis of solution of equilibrium equation. There raises the question, when it is possible to use differential equations (2) or (10) in the solution of space problem.

Solution of the task

Determination of components of stress tensor

The peculiarity of this solution is the use of combination of flat functions for components of stress

tensor, which is determined by the requiring of the solution of closed task of plasticity theory. This is explained under the condition when the solution satisfies all the system of equations of plasticity theory in the form of (1)...(5), qualitatively and quantitatively correct characterizes transition areas of plastic zones in various processes of metal treatment under pressure.

According to works [4]...[7], [12] and (7)...(11), there used trigonometric and fundamental substitutions of the following form

$$\tau_{xy} = k_1 \cdot \text{Sin}(A_1 F_1), \tau_{yz} = k_2 \cdot \text{Sin}(A_2 F_2), \tau_{zx} = k_3 \cdot \text{Sin}(A_3 F_3),$$

$$k_1 = C_{\sigma 1} \cdot \exp(\theta'_1), k_2 = C_{\sigma 2} \cdot \exp(\theta'_2), k_3 = C_{\sigma 3} \cdot \exp(\theta'_3), \quad (12)$$

Where  $\theta_3$  - are the arguments of goniometric functions depending on the coordinates  $X, Y, Z$ , defined by the task solution;

$\theta_3$  - are the exponents depending on the coordinates  $X, Y, Z$ , defined by the task solution.

During substitution (12) into (10), we will obtain the system of equations

The first equation

$$\left[ \theta'_{1xx} + (\theta'_{1x} + A_1 F_{1y})^2 - \theta'_{1yy} - (\theta'_{1y} - A_1 F_{1x})^2 \right] \cdot \text{Sin} A_1 F_1 +$$

$$+ \left[ 2 \cdot (\theta'_{1x} + A_1 F_{1y}) \cdot (A_1 F_{1x} - \theta'_{1y}) + A_1 F_{1xx} - A_1 F_{1yy} \right] \cdot \text{Cos} A_1 F_1 =$$

$$= -2 \cdot A_1 F_{1xy} \cdot \text{Sin} A_1 F_1 + 2 \cdot \theta'_{1xy} \cdot \text{Cos} A_1 F_1,$$

The second equation

$$\left[ \theta'_{2yy} + (\theta'_{2y} + A_2 F_{2z})^2 - \theta'_{2zz} - (\theta'_{2z} - A_2 F_{2y})^2 \right] \cdot \text{Sin} A_2 F_2 +$$

$$+ \left[ 2 \cdot (\theta'_{2y} + A_2 F_{2z}) \cdot (A_2 F_{2y} - \theta'_{2z}) + A_2 F_{2yy} - A_2 F_{2zz} \right] \cdot \text{Cos} A_2 F_2 = \quad (13)$$

$$= -2 \cdot A_2 F_{2yz} \cdot \text{Sin} A_2 F_2 + 2 \cdot \theta'_{2yz} \cdot \text{Cos} A_2 F_2,$$

The third equation

$$\begin{aligned} & \left[ \theta'_{3zz} + (\theta'_{3z} + A_3 F_{3x})^2 - \theta'_{3xx} - (\theta'_{3x} - A_3 F_{3z})^2 \right] \cdot \text{Sin} A_3 F_3 + \\ & + \left[ 2 \cdot (\theta'_{3z} + A_3 F_{3x}) \cdot (A_3 F_{3z} - \theta'_{3x}) + A_3 F_{3zz} - A_3 F_{3xx} \right] \cdot \text{Cos} A_3 F_3 = \\ & = -2 \cdot A_3 F_{3zx} \cdot \text{Sin} A_3 F_3 + 2 \cdot \theta'_{3zx} \cdot \text{Cos} A_3 F_3 \end{aligned}$$

In the equations (13) there appear operators before similar brackets. The solution runs forward when the goniometric functions. In each operator there appear brackets

$$\begin{aligned} & (\theta'_{1x} + A_1 F_{1y}), (\theta'_{1y} - A_1 F_{1x}), (\theta'_{2y} + A_2 F_{2z}), \\ & (\theta'_{3z} + A_3 F_{3x}), (\theta'_{3z} + A_3 F_{3x}), \theta'_{1x} = -A_1 F_{1y} \end{aligned}$$

are equal to zero, the equations turn into identical equations. Let us show this on the example of the first equation of the system (13)

$$\theta'_{1x} = -A_1 F_{1y}, \theta'_{1y} = A_1 F_{1x} \quad (14)$$

The second differential coefficients

$$\theta'_{1xx} = -A_1 F_{1yx}, \theta'_{1yy} = A_1 F_{1xy}, \theta'_{1xy} = -A_1 F_{1yy} = A_1 F_{1xx} \quad (15)$$

From the correlations (14) and (15) we come to the Laplace equations, which are satisfied by the functions  $\theta_1$  and  $A_1 F_1$

$$\theta'_{1xx} + \theta'_{1yy} = 0, A_1 F_{1xx} + A_1 F_{1yy} = 0 \dots \quad (16)$$

It is defined the class of added into review functions – they are balanced. Under such statement they are considered to be known. From the other point of view, correlations (14) and (15) turn (13) into identical equations. In such a way, functions  $\tau$  may be solved as follows

$$\begin{aligned} \tau_{xy} &= C_{\sigma_1} \cdot \exp \theta_1' \cdot \text{Sin} A_1 F_1, \quad \tau_{yz} = C_{\sigma_2} \cdot \exp \theta_2' \cdot \text{Sin} A_2 F_2, \\ \tau_{zx} &= C_{\sigma_3} \cdot \exp \theta_3' \cdot \text{Sin} A_3 F_3, \end{aligned} \quad (17)$$

when

$$\theta'_{1x} = -A_1 F_{1y}, \theta'_{1y} = A_1 F_{1x}; \theta'_{2y} = -A_2 F_{2z}, \theta'_{2z} = A_2 F_{2y};$$

$$\theta'_{3z} = -A_3 F_{3x}, \theta'_{3x} = A_3 F_{3z};$$

$$\theta'_{1xx} + \theta'_{1yy} = 0, A_1 F_{1xx} + A_1 F_{1yy} = 0;$$

$$\theta'_{2yy} + \theta'_{2zz} = 0, A_2 F_{2yy} + A_2 F_{2zz} = 0;$$

$$\theta'_{3zz} + \theta'_{3xx} = 0, A_3 F_{3zz} + A_2 F_{3xx} = 0.$$

Normal stresses  $\sigma_x, \sigma_y, \sigma_z$  are determined according to the known tangential stresses (17) from

the equilibrium equation. Taking into account deviatoric component we will have

$$\begin{aligned} \sigma'_x &= k_1 \cdot \text{Cos}(A_1 F_1) - k_3 \cdot \text{Cos}(A_3 F_3) + \sigma_0 + f(y, z) + C, \\ \sigma'_y &= -k_1 \cdot \text{Cos}(A_1 F_1) + k_2 \cdot \text{Cos}(A_2 F_2) + \sigma_0 + f(x, z) + C, \end{aligned} \quad (18)$$

$$\sigma'_y - \sigma'_z = 2 \cdot k_2 \cdot \text{Cos}(A_2 F_2) - k_3 \cdot \text{Cos}(A_3 F_3) - k_1 \cdot \text{Cos}(A_1 F_1).$$

Using (18) let us return to the difference of normal stresses in equations (8)

$$\begin{aligned} \sigma'_y - \sigma'_z &= 2 \cdot k_2 \cdot \text{Cos}(A_2 F_2) - k_3 \cdot \text{Cos}(A_3 F_3) - k_1 \cdot \text{Cos}(A_1 F_1), \\ \sigma'_y - \sigma'_z &= 2 \cdot k_2 \cdot \text{Cos}(A_2 F_2) - k_3 \cdot \text{Cos}(A_3 F_3) - k_1 \cdot \text{Cos}(A_1 F_1), \\ \sigma'_z - \sigma'_x &= 2 \cdot k_3 \cdot \text{Cos}(A_3 F_3) - k_1 \cdot \text{Cos}(A_1 F_1) - k_2 \cdot \text{Cos}(A_2 F_2). \end{aligned} \quad (19)$$

The given differences represent the combination of flat functions, more complicated than (9). It is possible to show that mixed derivatives from the differences in (9) and in (19) give the same result and that is why it acceptable to use flat functions for stresses. Let us show it:

$$\begin{aligned} \text{Plane problem} \quad \frac{\partial}{\partial x \partial y} 2 \cdot \tau_{xy} F_1 &= 2 \cdot \frac{\partial}{\partial x \partial y} k_1 \text{Cos} A_1 F_1 = 2(k_{1xy} - k_1 \cdot A_1 F_{1x} \cdot A_1 F_{1y}) \cdot \text{Cos} A_1 F_1 - \\ &- 2(k_1 A_1 F_{1xy} + k_{1xy} A_1 F_{1x} + k_{1xy} A_1 F_{1y}) \cdot \text{Sin} A_1 F_1, \end{aligned}$$

Space problem

$$\begin{aligned} &= 2(k_{1xy} - k_1 \cdot A_1 F_{1x} \cdot A_1 F_{1y}) \cdot \text{Cos} A_1 F_1 - 2(k_1 A_1 F_{1xy} + k_{1xy} A_1 F_{1x} + \\ &= 2(k_{1xy} - k_1 \cdot A_1 F_{1x} \cdot A_1 F_{1y}) \cdot \text{Cos} A_1 F_1 - 2(k_1 A_1 F_{1xy} + k_{1xy} A_1 F_{1x} + \\ &\quad + k_{1xy} A_1 F_{1y}) \text{Sin} A_1 F_1 \end{aligned}$$

Shown peculiarity is the principal detail in the presented solution. Taking into account the operator of the exponent, the following solutions are possible. Stress tensor components of space problem

$$\begin{aligned} \sigma'_x &= \pm C_{\sigma_1} \cdot \exp \theta'_1 \cdot \text{Cos}(A_1 F_1) \mp C_{\sigma_3} \cdot \exp \theta'_3 \cdot \text{Cos}(A_3 F_3) + \sigma_0 + C, \\ \sigma'_y &= \mp C_{\sigma_1} \cdot \exp \theta'_1 \cdot \text{Cos}(A_1 F_1) \pm C_{\sigma_2} \cdot \exp \theta'_2 \cdot \text{Cos}(A_2 F_2) + \sigma_0 + C, \\ \sigma'_z &= \pm C_{\sigma_3} \cdot \exp \theta'_3 \cdot \text{Cos}(A_3 F_3) \mp C_{\sigma_2} \cdot \exp \theta'_2 \cdot \text{Cos}(A_2 F_2) + \sigma_0 + C, \\ \tau_{xy} &= C_{\sigma_1} \cdot \exp \theta'_1 \cdot \text{Sin} A_1 F_1, \tau_{yz} = C_{\sigma_2} \cdot \exp \theta'_2 \cdot \text{Sin} A_2 F_2, \\ \tau_{zx} &= C_{\sigma_3} \cdot \exp \theta'_3 \cdot \text{Sin} A_3 F_3, \end{aligned} \quad (20)$$

when

$$\begin{aligned} \theta'_{1x} &= \mp A_1 F_{1y}, \quad \theta'_{1y} = \pm A_1 F_{1x}; \quad \theta'_{2y} = \mp A_2 F_{2z}, \quad \theta'_{2z} = \pm A_2 F_{2y}; \\ \theta'_{3z} &= \mp A_3 F_{3x}, \quad \theta'_{3x} = \pm A_3 F_{3z}; \\ \theta'_{1xx} + \theta'_{1yy} &= 0, \quad A_1 F_{1xx} + A_1 F_{1yy} = 0; \\ \theta'_{2yy} + \theta'_{2zz} &= 0, \quad A_2 F_{2yy} + A_2 F_{2zz} = 0; \\ \theta'_{3zz} + \theta'_{3xx} &= 0, \quad A_3 F_{3zz} + A_3 F_{3xx} = 0. \end{aligned}$$

**Determination of components of rate of deformation tensor**

Kinematic part of the task is a peculiar kind of test for the stationary one, as through the equation of connection there formed restrictions on stress functions. To satisfy the second part of the system of differential equations (4b), the simplest variant is the one, where

$$\frac{2 \cdot k_1 \cdot \text{Cos}(A_1 F_1) - k_2 \cdot \text{Cos}(A_2 F_2) - k_3 \cdot \text{Cos}(A_3 F_3)}{2 \cdot k_1 \cdot \text{Sin}(A_1 F_1)} = \frac{\xi_x - \xi_y}{\gamma_{xy}}$$

From the last expression it follows that the speed of deformations in the right part, under certain cor-

related derivatives from shear speed of deformations in the left parts equal zero. Acceptable are:

$\gamma_{xy} = f(x, y)$ ;  $\gamma_{yz} = f(y, z)$   $\gamma_{zx} = f(z, x)$ , linear speeds of deformations and normal stresses as well, represent the combination of flat functions. The first equation of connection (3) will be as follows

$$\xi_x = \beta_1 \cdot \text{Cos}(B_1 F_1) - \beta_3 \cdot \text{Cos}(B_3 F_3), \xi_y = -\beta_1 \cdot \text{Cos}(B_1 F_1) + \beta_2 \cdot \text{Cos}(B_2 F_2),$$

$$\gamma_{xy} = 2 \cdot \beta_1 \cdot \text{Sin}(B_1 F_1)$$

Herein

$$\frac{\xi_x - \xi_y}{\gamma_{xy}} = \frac{2 \cdot \beta_1 \cdot \text{Cos}(B_1 F_1) - \beta_2 \cdot \text{Cos}(B_2 F_2) - \beta_3 \cdot \text{Cos}(B_3 F_3)}{2 \cdot \beta_1 \cdot \text{Sin}(B_1 F_1)}$$

Qualitative solution for deformation rate in general looks like

$$\begin{aligned} \xi_x &= \pm \beta_1 \cdot \text{Cos}(B_1 F_1) \mp \beta_3 \cdot \text{Cos}(B_3 F_3), \\ \xi_y &= \pm \beta_2 \cdot \text{Cos}(B_2 F_2) \mp \beta_1 \cdot \text{Cos}(B_1 F_1), \\ \xi_z &= \pm \beta_3 \cdot \text{Cos}(B_3 F_3) \mp \beta_2 \cdot \text{Cos}(B_2 F_2). \end{aligned} \tag{21}$$

$$\gamma_{xy} = 2 \cdot \beta_1 \cdot \text{Sin} B_1 F_1, \gamma_{yz} = 2 \cdot \beta_2 \cdot \text{Sin} B_2 F_2, \gamma_{zx} = 2 \cdot \beta_3 \cdot \text{Sin} B_3 F_3.$$

The condition of volume constancy is satisfied

$$\xi_x + \xi_y + \xi_z = 0$$

In the equations of connection the arguments of goniometric functions have the same functions F. Herein the functions with index 1 are determined by the coordinates XY, 2 - YZ, 3 - ZX. In the formulas (21)  $\beta_1, \beta_2, \beta_3$  are the unknown variables. Let us

substitute deformation rates (21) into the first three differential continuity equations of deformation rates (4a), than after reduction and simplifications the equations come to

$$\begin{aligned} \frac{\partial^2 \beta_1 \cdot \text{Cos}(A_1 F_1)}{\partial y^2} - \frac{\partial^2 \beta_1 \text{Cos}(A_1 F_1)}{\partial x^2} &= 2 \cdot \frac{\partial^2 \beta_1 \cdot \text{Sin}(A_1 F_1)}{\partial y \partial x}, \\ \frac{\partial^2 \beta_2 \cdot \text{Cos}(A_2 F_2)}{\partial z^2} - \frac{\partial^2 \beta_2 \cdot \text{Cos}(A_2 F_2)}{\partial y^2} &= 2 \cdot \frac{\partial^2 \beta_2 \cdot \text{Sin}(A_2 F_2)}{\partial z \partial y}, \\ \frac{\partial^2 \beta_3 \cdot \text{Cos}(A_3 F_3)}{\partial x^2} - \frac{\partial^2 \beta_3 \cdot \text{Cos}(A_3 F_3)}{\partial z^2} &= 2 \cdot \frac{\partial^2 \beta_3 \cdot \text{Sin}(A_3 F_3)}{\partial x \partial z} \end{aligned} \tag{22}$$

Despite the complex combination of functions in the equations of the system (22) there are similar unknown variables  $\beta$ , enter the system of equations in

the 1<sup>st</sup> degree. This allows to use fundamental substitution as follows [12]:

$$\beta_1 = C_{\xi_1} \cdot \exp \theta_1'', \quad \beta_2 = C_{\xi_2} \cdot \exp \theta_2'', \quad \beta_3 = C_{\xi_3} \cdot \exp \theta_3'' \quad (23)$$

Substituting (23) into (22) we will obtain the system

The first equation

$$\begin{aligned} & \left[ -\theta_{1xx}'' - (\theta_{1x}'' + B_1 F_{1y})^2 + \theta_{1yy}'' + (\theta_{1y}'' - B_1 F_{1x})^2 \right] \cdot \text{Cos} B_1 F_1 + \\ & + \left[ 2 \cdot (B_1 F_{1x} - \theta_{1y}'') \cdot (\theta_{1x}'' + B_1 F_{1y}) + (B_1 F_{1xx} - B_1 F_{1yy}) \right] \cdot \text{Sin} B_1 F_1 = \\ & = 2 \cdot B_1 F_{1xy} \cdot \text{Cos} B_1 F_1 + 2 \cdot \theta_{1xy}'' \cdot \text{Sin} B_1 F_1, \end{aligned}$$

The second equation

$$\begin{aligned} & \left[ -\theta_{2yy}'' - (\theta_{2y}'' + B_2 F_{2z})^2 + \theta_{2zz}'' + (\theta_{2z}'' - B_2 F_{2y})^2 \right] \cdot \text{Cos} B_2 F_2 + \\ & + \left[ 2 \cdot (B_2 F_{2y} - \theta_{2z}'') \cdot (\theta_{2y}'' + B_2 F_{2z}) + (B_2 F_{2yy} - B_2 F_{2zz}) \right] \cdot \text{Sin} B_2 F_2 = \\ & = 2 \cdot B_2 F_{2yz} \cdot \text{Cos} B_2 F_2 + 2 \cdot \theta_{2yz}'' \cdot \text{Sin} B_2 F_2, \end{aligned} \quad (24)$$

The third equation

$$\begin{aligned} & \left[ -\theta_{3zz}'' - (\theta_{3z}'' + B_3 F_{3x})^2 + \theta_{3xx}'' + (\theta_{3x}'' - B_3 F_{3z})^2 \right] \cdot \text{Cos} B_3 F_3 + \\ & + \left[ 2 \cdot (B_3 F_{3z} - \theta_{3x}'') \cdot (\theta_{3z}'' + B_3 F_{3x}) + (B_3 F_{3zz} - B_3 F_{3xx}) \right] \cdot \text{Sin} B_3 F_3 = \\ & = 2 \cdot B_3 F_{3zx} \cdot \text{Cos} B_3 F_3 + 2 \cdot \theta_{3zx}'' \cdot \text{Sin} B_3 F_3. \end{aligned}$$

As in the case of (13) the system (24) may be solved if there is a possibility to avoid the nonlinear effect. Let us taking the brackets in operators equal zero, than

$$\begin{aligned} \theta_{1x}'' = -B_1 F_{1y}, \quad \theta_{1y}'' = B_1 F_{1x}, \quad \theta_{2z}'' = B_2 F_{2y}, \quad \theta_{2y}'' = B_2 F_{2z}, \\ \theta_{3z}'' = -B_3 F_{3x}, \quad \theta_{3x}'' = -B_1 F_{1y}. \end{aligned} \quad (25)$$

The second differential coefficients

$$\theta_{1x}'' = -B_1 F_{1y}, \quad \theta_{1y}'' = B_1 F_{1x}; \quad \theta_{2y}'' = -B_2 F_{2z}, \quad \theta_{2z}'' = B_2 F_{2y};$$

$$\theta_{3z}'' = -B_3 F_{3x}, \quad \theta_{3x}'' = B_3 F_{3z};$$

$$\theta_{1xx}'' + \theta_{1yy}'' = 0, \quad B_1 F_{1xx} + B_1 F_{1yy} = 0;$$

$$\theta_{2yy}'' + \theta_{2zz}'' = 0, \quad B_2 F_{2yy} + B_2 F_{2zz} = 0;$$

$$\theta_{3zz}'' + \theta_{3xx}'' = 0, \quad B_3 F_{3zz} + B_3 F_{3xx} = 0.$$

Input functions  $\theta_i''$  and  $B_i F_i$  are the harmonic ones, and the functions for stress tensor components as well. The expression for determination tensor components of deformation speed for space task, taking into account (21)...(25), looks as follows

$$\begin{aligned} \xi_x &= \pm \beta_1 \cdot \text{Cos}(B_1 F_1) \mp \beta_3 \cdot \text{Cos}(B_3 F_3), \\ \xi_y &= \pm \beta_2 \cdot \text{Cos}(B_2 F_2) \mp \beta_1 \cdot \text{Cos}(B_1 F_1), \\ \xi_z &= \pm \beta_3 \cdot \text{Cos}(B_3 F_3) \mp \beta_2 \cdot \text{Cos}(B_2 F_2) \end{aligned} \quad (26)$$

$$\gamma_{xy} = 2 \cdot C_{\xi_1} \cdot \exp \theta_1'' \cdot \sin B_1 F_1, \quad \gamma_{yz} = 2 \cdot C_{\xi_2} \cdot \exp \theta_2'' \cdot \sin B_2 F_2,$$

$$\gamma_{zx} = 2 \cdot C_{\xi_3} \cdot \exp \theta_3'' \cdot \sin B_3 F_3,$$

when

$$\theta_{1x}'' = \mp B_1 F_{1y}, \quad \theta_{1y}'' = \pm B_1 F_{1x}; \quad \theta_{2y}'' = \mp B_2 F_{2z}, \quad \theta_{2z}'' = \pm B_2 F_{2y};$$

$$\theta_{3z}'' = \mp B_3 F_{3x}, \quad \theta_{3x}'' = \pm B_3 F_{3z};$$

$$\theta_{1xx}'' + \theta_{1yy}'' = 0, \quad B_1 F_{1xx} + B_1 F_{1yy} = 0;$$

$$\theta_{2yy}'' + \theta_{2zz}'' = 0, \quad B_2 F_{2yy} + B_2 F_{2zz} = 0;$$

$$\theta_{3zz}'' + \theta_{3xx}'' = 0, \quad B_3 F_{3zz} + B_3 F_{3xx} = 0.$$

**Integral characteristic of stress state**

From the above mentioned it follows that the closed set and solution of the task of plasticity theory does not complicate it, and at certain approaches – simplifies and allows to find the solution of space

problem in analytical form. To find generalized indexes of stress state of the point with the use of solution (22) is of some interest. Let us consider some variants represented by correlations (22). Let us

$$\sigma'_x = k_1 \cdot \cos(A_1 F_1) - k_3 \cdot \cos(A_3 F_3) + \sigma_0 + f(y, z) + C,$$

$$\sigma'_y = -k_1 \cdot \cos(A_1 F_1) + k_2 \cdot \cos(A_2 F_2) + \sigma_0 + f(x, z) + C,$$

$$\sigma'_z = k_3 \cdot \cos(A_3 F_3) - k_2 \cdot \cos(A_2 F_2) + \sigma_0 + f(x, y) + C.$$

Differences of normal stresses

$$\sigma'_x - \sigma'_y = 2 \cdot k_1 \cdot \cos(A_1 F_1) - k_2 \cdot \cos(A_2 F_2) - k_3 \cdot \cos(A_3 F_3),$$

$$\sigma'_y - \sigma'_z = 2 \cdot k_2 \cdot \cos(A_2 F_2) - k_3 \cdot \cos(A_3 F_3) - k_1 \cdot \cos(A_1 F_1),$$

$$\sigma'_i = \sqrt{3} \cdot \sqrt{k_1^2 + k_2^2 + k_3^2 - k_1 \cos A_1 F_1 \cdot k_3 \cdot \cos A_3 F_3 - k_1 \cos A_1 F_1 \cdot k_2 \cdot \cos A_2 F_2 -$$

Substituting into (1) we will obtain

$$\sigma'_i = \sqrt{3} \cdot \sqrt{k_1^2 + k_2^2 + k_3^2 - k_1 \cos A_1 F_1 \cdot k_3 \cdot \cos A_3 F_3 - k_1 \cos A_1 F_1 \cdot k_2 \cdot \cos A_2 F_2 - k_3 \cos A_3 F_3 \cdot k_2 \cdot \cos A_2 F_2} \quad (27)$$

It should be mentioned that under the radical there is the sum of squares of flat functions for values  $k$ . Appearance of difference of goniometric functions makes significant allowances to the result and affects

the integral characteristic of trigonometric component. At  $k_2 = 0$  the expression (27) is simplified and looks as follows

$$\sigma'_i = \sqrt{3} \cdot \sqrt{k_1^2 + k_3^2 - k_1 \cos A_1 \Phi_1 \cdot k_3 \cdot \cos A_3 \Phi_3}$$

If  $k_2 = k_3 = 0$ , we have a flat task of plasticity theory, than

$$\sigma'_i = \sqrt{3} \cdot \sqrt{k_1^2} = \sqrt{3} \cdot k_1 = \sigma_T, \quad \text{t.e.} \quad k_1 = \frac{\sigma_T}{\sqrt{3}}$$

Variant when all the stresses may be shifted to the negative zone on the account of mean stress  $\sigma_0$  looks as follows

$$\sigma_x'' = -k_1' \cdot \text{Cos}(A_1 F_1) + k_3 \cdot \text{Cos}(A_3 F_3) + \sigma_0 + C,$$

$$\sigma_y'' = +k_1 \cdot \text{Cos}(A_1 F_1) + k_2 \cdot \text{Cos}(A_2 F_2) + \sigma_0 + C,$$

$$\sigma_z'' = -k_3 \cdot \text{Cos}(A_3 F_3) - k_2 \cdot \text{Cos}(A_2 F_2) + \sigma_0 + C,$$

Difference of normal stresses

$$\sigma_x'' - \sigma_y'' = -2 \cdot k_1 \cdot \text{Cos}A_1 F_1 + k_3 \cdot \text{Cos}A_3 F_3 - k_2 \cdot \text{Cos}A_2 F_2,$$

$$\sigma_y'' - \sigma_z'' = 2 \cdot k_2 \cdot \text{Cos}A_2 F_2 + k_3 \cdot \text{Cos}A_3 F_3 + k_1 \cdot \text{Cos}A_1 F_1,$$

$$\sigma_z'' - \sigma_x'' = -2 \cdot k_3 \cdot \text{Cos}A_3 F_3 + k_1 \cdot \text{Cos}A_1 F_1 - k_2 \cdot \text{Cos}A_2 F_2.$$

Substituting into the stress intensity, we will obtain

$$\sigma_i'' = \sqrt{3} \cdot \sqrt{k_1^2 + k_2^2 + k_3^2 - k_1 \text{Cos}A_1 F_1 \cdot k_3 \cdot \text{Cos}A_3 F_3 + k_1 \text{Cos}A_1 F_1 \cdot k_2 \cdot \text{Cos}A_2 F_2 + k_3 \text{Cos}A_3 F_3 \cdot k_2 \cdot \text{Cos}A_2 F_2} \quad (28)$$

The sum of squares of values  $k$  remains unchanged. When compare expressions (27) and (28), one may see that the operators before goniometric functions change. This lead to the changes in calculations. In case of (27), intensity may be greater than in case of (28). Stability of sum of squares  $k$  at various combinations of stresses testifies that the intensity of

stresses changes near this value to the bigger or smaller site and value

$$\sigma_i'' = \sqrt{3} \cdot \sqrt{k_1^2 + k_2^2 + k_3^2} = \sqrt{3} \cdot k,$$

is a peculiar kind of core of solution, At  $k_1 = 0$  the expression (28) is simplified and turns to

$$\sigma_i'' = \sqrt{3} \cdot \sqrt{k_2^2 + k_3^2 + k_2 \text{Cos}A_2 \Phi_2 \cdot k_3 \cdot \text{Cos}A_3 \Phi_3}$$

If  $k_1 = k_2 = 0$ , we will obtain the flat task of plasticity theory, than

$$\sigma_i'' = \sqrt{3} \cdot \sqrt{k_3^2} = \sqrt{3} \cdot k_3 = \sigma_T, \text{ т.е. } k_3 = \frac{\sigma_T}{\sqrt{3}}.$$

From the analysis of integral characteristics of stress state, it is seen that the result of space task, during simplifying, may be compared with known dependences for flat task. This allows to consider obtained solution at volume loading as more general and more complex characteristic of stress state of the point, satisfying the closed system of equations of plasticity theory.

### Conclusions

1. Integral characteristics of stress state determine mechanical characteristics of plastic medium. Presented correlation may be used for account of nonuniformity of mechanical properties of processed material.
2. There set closed volume task of plasticity theory (static and kinematic part) allowing on some extent not to complicate the solution but to simplify it.
3. Correction factor of simple solution is kinematic part of the task, which is easier to satisfy

by the combinations of flat functions.

4. The obtained result is in some correspondence with solution of flat task and may be recommended for determination of components of stress tensor, deformation tensor and generalized indexes of point state.

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## Analysis and modelling of complex rheologic mediums in conditions of thermomechanical loading

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