Analysis of Chaotic Runoff Data Based on Chebyshev Polynomials Local Model

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Abstract
In this paper, we analyze dynamics of runoff data from actual hydrologic station on the basis of the chaos theory. Through analyzing Characteristics of power spectrum, geometric structure of attractors and calculating Lyapunov index, it confirmed that actual runoff has chaos dynamics behavior. Local prediction model based on chaotic runoff data is inferior for heavy computation load and poor accuracy under high order of polynomial and big sample size. To address these problems, combined the phase-space reconstruction technology, we propose a chaos local prediction model for runoff data based on Chebyshev Polynomials. Integrating the advantage of approximation effect of Chebyshev Polynomials with the nonlinear prediction ability of local prediction model, the proposed model could express dynamic characteristics of hydrologic runoff system well, increases prediction accuracy and enhances anti-noise performance of the model. Result of simulation shows that the model still gets relatively satisfying prediction effect under data pollution by white noise.
Key words: CHAOS, CHEBYSHEV POLYNOMIALS, RUNOFF, LOCAL PREDICTION MODEL

1. Introduction

Hydrologic system is a complex system and runoff is vulnerable to astronomy, climate and human activities. Abundant empirical researches reported that runoff is rich of chaotic behaviors [1-8]. Therefore, prediction method based on the chaos theory is better to study problems concerning runoff forecasting [9-11].

Chaos looks like an irregular movement. It is the random-like behaviors occurred in a deterministic complex system without adding any random factors. Chaos degree of one system could be depicted by the maximum Lyapunov index λ. If $\lambda > 0$, it is a chaotic runoff sequence. Higher λ represents that the chaotic degree of runoff is higher [12-13].

In the prediction field of chaotic time series, the local prediction model based on phase-space reconstruction attracts increasing attentions due to its good approximation function [12-17]. However, under high order of polynomial and big data size, the local model has to deal with heavy computation load. Due to the strict restriction of round-off error, the prediction result is often unstable and it even will produce meaningless numerical results [12-13]. Therefore, this paper proposed a chaos local prediction model of runoff based on Chebyshev polynomials. Combining the good mathematical properties of Chebyshev polynomials and strong nonlinear prediction capability of local models, the proposed model achieves higher prediction accuracy and noise resistance.

Chebyshev polynomials have a series of good mathematical properties like quick convergence and good approximation results [18]. It not only plays an important role in numerical approximation, but also could be applied in prediction of chaotic time series.

2. Chebyshev polynomials

The weight function ($w(t)=1$) constitutes the orthogonal polynomials ($P_n(t)\ (n=0,1,\cdots)$) on the interval $[-1,1]$. It is called as L multiform and its expression is $T_n(t)=\cos(n\arccos x), -1\le t\le 1$. The Chebyshev polynomials have some important properties:

(1) Orthogonality

$$(T_n,T_m)=\int_{-1}^{1}\frac{1}{\sqrt{1-xt^2}}T_n(t)T_m(t)dt=\begin{cases}0, & n\neq m\\ \pi/2, & m=n\neq 0\\ \pi, & m=m=0\end{cases}$$

Odevity: $T_n(-t)=(-1)^nT_n(t)$

Maximum character: $\max\limits_{-1\le x\le 1}|T_n(t)|=1$.

Make linear transformation $t=\frac{2x-(a+b)}{b-a}$ and the Chebyshev polynomials within the interval $[a,b]$ could be gained.

$$T_n(x)=\cos(n\arccos\frac{2x-a-b}{b-a}),\ a\le x\le b$$

It is easy to verify that the Chebyshev polynomials within the interval $[a,b]$ has the following recursive relation:

$$T_{n+1}(x)=\frac{2[2x-(a+b)]}{b-a}T_n(x)-T_{n-1}(x)$$

Then, the following expressions of different orders of Chebyshev polynomials could be gained:

$$T_0(x)=1$$

$$T_1(x)=\frac{2x-(a+b)}{b-a}$$

$$T_2(x)=2\left[\frac{2x-(a+b)}{b-a}\right]^2-1$$

$$T_3(x)=4\left[\frac{2x-(a+b)}{b-a}\right]^3-3\left[\frac{2x-(a+b)}{b-a}\right]$$

$$\cdots\ \cdots\ \cdots$$

Chebyshev polynomial, a very important polynomial, has a series of excellent properties like orthogonality and quick convergence. The Chebyshev function has quick approximation convergence and could achieve high accuracy of numerical approximation to point of view only with the first few items.

There's an approximation theorem of Chebyshev polynomials [18]: if $\{T_i(x)\}|_{i=0}^{n}$ is the Chebyshev polynomials sequence on the interval $[a,b]$, $f\in C[a,b]$ and

$$\Phi=span\{T_{0,}T_{1,},\cdots,T_{n,}\}=\{T(x)\,|\,T(x)=\sum_{k=0}^{n}a_kT_k(x),a_k\in R\}$$

Then, there's a recent uniform approximation $T^*(x)\in\Phi$ that makes $\|f-T^*\|=\inf\limits_{T\in\Phi}\|f-T\|$.

3. Local prediction model

3.1 Phase-space reconstruction theory

Dynamic prediction and statistical prediction are two traditional methods to forecast runoff time sequence. Since hydrological runoff is influenced by many factors, the dynamic prediction is difficult to get a comprehensive understanding on the physical mechanism of actual runoff due to the ambiguous relationship between the law of motion of hydrological runoff system and influencing factors as well as even incomplete or missing data of influencing factors. Many of existing prediction models only considers linear factors of runoff, but neglect the nonlinear behavior characteristics. Abundant researches reported that such prediction model has a shortage of runoff evolution data missing. To solve this problem, we used the phase-space reconstruction technology proposed by Takens to disclose the geometric structure of phase space of complicated hydrological runoff and then established the prediction model of runoff time series. The theoretical basis of phase-space reconstruction is the following Takens theorem [19].

Takens theorem: Let $\{x(k)\}_{k=1,2,\cdots,n}$ be a time series. Given the delay time τ and embedding dimension m, a smooth map F could be found on the attractor: $R^m \to R^m$, which meets $Y(t+1) = F[Y(t)]$. $Y(n)$ is the *m*-dimensional vector:

$$Y(t) = (x(t), x(t-\tau), \cdots, x(t-(m-1)\tau),$$
$$t = 1, 2, \cdots, n-(m-1)\tau$$

According to the Takens theorem, orbits in the reconstructed phase space is equivalent to the original chaotic system in the sense of diffeomorphism. Therefore, the reconstructed phase space could maintain same geometrical characteristics and information with the original dynamical system. This enables us to study dynamic behaviors and features of the system with some known information even though we haven't a comprehensive understanding on the correlation of variables in the chaotic system.

3.2 Modeling

Suppose $\{x(k)\}_{k=1,2,\cdots,n}$ is a known runoff time series. We will predict this time series, that is, predicting its future evolution state: $x(n+1), x(n+2), \cdots$.

Firstly, reconstruct a phase space for this time series. It can be known from the Takens theorem that choosing appropriate τ and m could get corresponding $N = n-(m-1)\tau$ phase points of this time series in the reconstructed *m*-dimensional phase space:

$$Y(1) = \left(x(1), x(1+\tau), \cdots, x(1+(m-1)\tau\right),$$
$$Y(2) = (x(2), x(2+\tau), \cdots, x(2+(m-1)\tau), \ldots,$$
$$Y(N) = \left(x(N), x(N+\tau), \cdots, x(n)\right).$$

There, $Y(N)$ is the prediction center. The key to predict $x(n+1), x(n+2), \cdots$ is to find out the mapping *F* that meets

$$Y(N+1) = F[Y(N)].$$

Since chaotic behaviors only occur under nonlinear *F*, we will use high-order polynomials as an approximation representation of *F*. The Weierstrass approximation theory states that any continuous function in any a small neighborhood field could be approximated by polynomials and achieve the expecting accuracy. Polynomials are numerical computation and are easy to be implemented on computer simulation.

Polynomials of the functional relationship *F* between $Y(N)$ and its next step $Y(N+1)$ could be expressed approximately as:

$$Y(N+1) = F[Y(N)] = b_0 + b_1 Y(N) + b_2 Y^2(N) + \cdots + b_q Y^q(N) \quad (1)$$

In fact, model (1) is a traditional prediction model of chaotic time series, which is called as a high-order local prediction model. With this model, we could get coefficients $b_0, b_1, \cdots, b_q$ by using some adjacent points to the prediction center $Y(N)$ as the reference points and then predict the point $Y(N+1)$ in the reconstructed phase space through historical data. In this way, the predicted value of time series $x(n+1)$ could be gained, because it is the last component of the phase point $Y(N+1)$. Repeat above steps and iterate continuously, and the $x(n+1), x(n+2), \cdots$ could be predicted. This is the idea of local prediction model of chaotic runoff time series based on phase-space reconstruction technology.

It is a pity that under high order (q) and big sample size, model (1) has to deal with heavy computation loads to get $b_0, b_1, \cdots, b_q$. These coefficients are closely correlated, which will influence the prediction accuracy of the model. To eliminate these phenomena, we used some good mathematical properties of Chebyshev polynomials (quick convergence, good approximation effect and orthogonality) to establish the high-order local prediction model of chaotic runoff time series.

Suppose the interval range of runoff is $[a,b]$. Now, the Chebyshev polynomial $\{T_n(x), n = 0, 1, 2, \cdots\}$ on [a,b] to replace the polynomial sequence $\{Y^n, n = 0, 1, 2, \cdots,\}$ in the model (1). Then,

$$Y(N+1) = \beta_0 + \beta_1 T_1\left[Y(N)\right] + \beta_2 T_2\left[Y(N)\right] + \cdots + \beta_q T_q\left[Y(N)\right] \quad (2)$$

We called the model (2) as the local prediction model based on Chebyshev polynomials. Model coefficients $\beta_0, \beta_1, \cdots \beta_q$ are calculated in the following text.

Let $Y(k_1), Y(k_2), \cdots, Y(k_M)$ be M adjacent points to the prediction center $Y(N)$. According to equation (2), they shall meet:

$$Y(k_i+1) = \beta_0 + \beta_1 T_1\left[Y(k_i)\right] + \beta_2 T_2\left[Y(k_i)\right] + \cdots + \beta_q T_q\left[Y(k_i)\right], \; i = 1, 2, \cdots, M \quad (3)$$

The goal is to get the predicted value of the last component of phase point $Y(N+1)$. Therefore, attentions shall be paid to the last component of phase points only when calculating coefficients. Based on equation (3), the last components of adjacent phase points shall meet:

$$x(k_i+1) = \beta_0 + \beta_1 T_j\left[x(k_i)\right] + \beta_2 T_j\left[x(k_i)\right] + \cdots + \beta_q T_j\left[x(k_i)\right], \quad i = 1, 2, \cdots, M \quad (4)$$

Now, we have to determine model coefficients $\beta_0, \beta_1, \cdots, \beta_q$. Different effects of $Y(k_1), Y(k_2), \cdots, Y(k_M)$ on the prediction center $Y(N)$ shall be considered during this process. Therefore, weight of different adjacent phase points is defined firstly. Suppose,

$d_i = \|Y(N) - Y(k_i)\|$

Let

$$w(k_i) = \frac{1/d_i}{\sum_{i=1}^{M} 1/d_i}.$$

Obviously, weight $w(k_i)$ depicted effect of adjacent point $Y(k_i)$ on $Y(N)$. $Y(k_i)$ Closer to the $Y(N)$ will affect the $Y(N)$ more significantly.

We view equation (4) as the multivariate function of $\beta_0, \beta_1, \cdots, \beta_q$ and take $w(k_i)$ as the weight of the adjacent point $Y(k_i)$. According to the idea of weighted least square method, let:

$$e(\beta_0, \beta_1, \cdots, \beta_q) = \sum_{i=1}^{M} w(k_i) \left\{ x(k_i + 1) - \sum_{j=0}^{q} \beta_j T_j \left[x(k_i) \right] \right\}^2$$

Coefficients $\beta_0, \beta_1, \cdots, \beta_q$ shall ensure the sum of square error $e(\beta_0, \beta_1, \cdots, \beta_q)$ be the minimum. Hence, let partial derivatives of $\beta_0, \beta_1, \cdots, \beta_q$ be zero, and then,

$$\sum_{m=0}^{q} \beta_m \sum_{i=1}^{M} w(k_i) \cdot T_m \left[x(k_i) \right] T_j [x(k_i)] = \sum_{i=1}^{M} w(k_i) \cdot x(k_i + 1) T_j [x(k_i)]$$

, $j = 0, 1, \cdots, q$

This is a normal equation set of $q+1$ unknown variables and $q+1$ equations. It could be simplified into a simple matrix expression:

$Y = TB$

Where

$$Y = \begin{bmatrix} w(k_1) \cdot x(k_1 + 1) \\ w(k_2) \cdot (k_2 + 1) \\ \cdots \cdots \cdots \\ w(k_M) \cdot x(k_M + 1) \end{bmatrix}, B = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{bmatrix}$$

$$T = \begin{bmatrix} w(k_1) & w(k_1) \cdot T_1 \left[x(k_1) \right] & \cdots & w(k_1) \cdot T_q \left[x(k_1) \right] \\ w(k_2) & w(k_2) \cdot T_1 \left[x(k_2) \right] & \cdots & w(k_2) \cdot T_q \left[x(k_2) \right] \\ & \cdots \cdots \cdots \cdots & & \\ w(k_M) & w(k_M) \cdot T_1 \left[x(k_M) \right] & \cdots & w(k_M) \cdot T_q \left[x(k_M) \right] \end{bmatrix}$$

Under T reversibility, the only one solution of the equation set could be gained:

$B = T^{-1} Y = (T'T)^{-1} T'Y$,

That is, coefficients in the model $\beta_0, \beta_1, \cdots \beta_q$ are acquired.

Now, the chaotic local prediction model (2) of runoff based on Chebyshev polynomials is established.

4. Numerical simulation

Chaotic behavioral characteristics of actual hydrological runoff are analyzed. On this basis, the model effectiveness is verified through a numerical simulation.

Data for numerical simulation are 1,096 daily runoff data from 2004 to 2006 recorded in the Wuzhou Hydrologic Station in Guangxi province. The corresponding time series are $\{x(t), t = 1, 2, \cdots, 1096\}$ (Figure1). These data will be used to establish the local prediction model to predict evolution trend of runoff time series. The prediction accuracy of this model will be tested.

4.1 Chaotic behavioral characteristics analysis of runoff

Whether this runoff time series have chaotic characteristics is judged by using power spectrum and phase diagram of attractors from the perspective of qualitative analysis.

Whether this runoff time series have chaotic forms of dynamical system could be judged intuitively from its power spectrum drawn based on Fourier analysis [12-13]. Power spectrum of chaotic time series has characteristics of continuity, noise background and broad peak. Power spectra of runoff series is shown in Figure2. All have continuity, noise background and broad peak, reflecting that they may have chaotic motion features. Under the logarithmic coordinates, obvious power law is observed on the power spectra. Low-frequency power spectra decline at the rate of power function. Power law is an external representation of chaotic features of runoff.

Attractor is an abstract mathematical model and the set of infinite points in phase space. Attractor of actual runoff series is shown in a 2D phase graph (Figure3). The attractor orbits of the runoff series gather together in the reconstructed 2D phase space, showing rough and irregular geometric structure and obvious binding characteristics. This confirms from another perspective that the runoff series may have chaotic dynamic characteristics [20].

The Lyapunov index of runoff series (λ) is calculated in the following text. Firstly, reconstruct the phase space for the runoff series. It can be known from the Takens theorem two important parameters, namely m and τ, are required.

Parameter τ is determined by autocorrelation function [20-21]. As shown in Figure 4, the x-coordinate is delay time and the y-coordinate is the autocorrelation function of the runoff series. When the autocorrelation function decreases to $1-\frac{1}{e}$ the delay, time τ is decided, and it calculated 13(Figure4). Parameter m is determined by Cao algorithm [22], as shown in Figure5, the x-coordinate is embedding dimension m and the y-coordinate is value of E1 and E2. We can see that E2 is floated up and down around 1, this result hints existing chaos in the runoff. It is also found that E1 attains its saturation value at

$m = 17$, and so does E2, therefore, 17 should be embedding dimension for this time series.

Next, Lyapunov index of the runoff time series could be calculated through small data sets method [23]. The separation velocity variation of two orbits in the phase space against time is shown in Figure6. Slope of the straight section is the value of the Lyapunov index λ, which is calculated larger than 0. These results give an intuitive representation that runoff motion is not simple periodic and random, but has qualitative feature of nonlinear dynamic evolution. This conforms to the previous intuitive judgment based on power spectra and geometric structure of attractors. Therefore, its evolution trend is predicted by using the proposed local model.

4.2 Prediction of runoff time series

The upper limit of predictable length is determined $T = 1/\lambda = 11$ from the Lyapunov index [20]. It means that the runoff volume in the next 11d could be predicted.

A total of 964 phase points are gained through phase-space reconstruction for these runoff time series. Among them, the $Y(964)$ is taken as the prediction center and its adjacent phase points are used as the fitting reference points. Chebyshev local prediction models when q=2 and 3 are established. The runoff time series are also predicted by using the traditional local linear model for the purpose of prediction effect comparison.

To evaluate prediction effect of models, relative error of prediction points is defined:

$$E = \frac{|x(t) - \hat{x}(t)|}{|x(t)|}$$

The regularization mean square error (MSE) for describing the overall prediction performance of the model is defined:

$$MSE = \frac{\sqrt{\sum_{t=1}^{N}\left[x(t) - \hat{x}(t)\right]^2}}{\sqrt{\sum_{t=1}^{N}\left[x(t) - \bar{x}(t)\right]^2}}.$$

Where, $\bar{x}(t)$ and $\hat{x}(t)$ are mean value and predicted value of the series.

Prediction effects are presented in Figure7. Obviously, predicted results of models are all close to the actual value of runoff. Table 1 lists the relative error of prediction models. Compared to traditional model, Chebyshev local model has smaller relative error, which implies this model takes better prediction performance compared by traditional local linear model.

Mean square errors of prediction of different models are listed in Table 2. Compared to traditional model, second-order and third-order Chebyshev models show significantly higher prediction accuracy. Although the third-order Chebyshev model has slightly higher prediction accuracy than the second-order one, no essential difference is observed between them. This means that we could achieve good approximation effect of runoff series by using the third-order orthogonal model.

Finally, noise resistance of prediction models are discussed. Noise resistance is an important performance of prediction models and is manifested by adaptability to noise under some noise threshold. A model with strong noise resistance shall not be too sensitive to noise and its coefficients shall keep stable to a certain extent before and after adding noises.

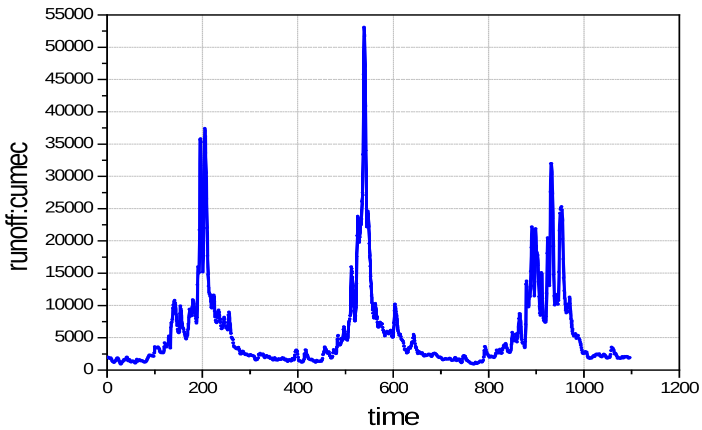

Figure 1. Runoff series

We add white noise sequence into the abovementioned runoff series to simulate noise in actual observed sequence. The noise power increases from 10^{-5} to 200. Establish prediction models for runoff series with noise and coefficient differences of models before and after noise adding are compared. Results are shown in Figure 8. The x-coordinate is noise power and the y-coordinate is error. As noise power increases, coefficients of models before and after noise adding increase accordingly. However, such change is more evident in the traditional model than other two orthogonal models. In other words, with the increase of noise power, coefficient errors of the traditional model change quickly. The model coefficient difference before and after noise adding is relatively distinct when the noise power increases to 0025. Such increase in the second-order and third-order models is relatively slowly. Therefore, we come to the conclusion that prediction models based on Chebyshev polynomials have stronger noise resistance compared to the traditional model.

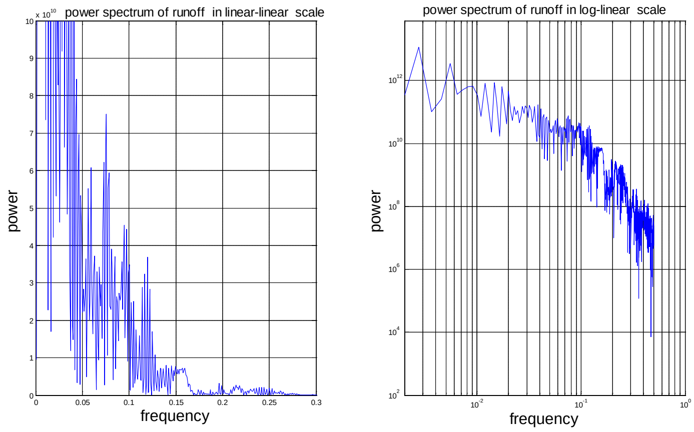

Figure 2. Power spectra of runoff

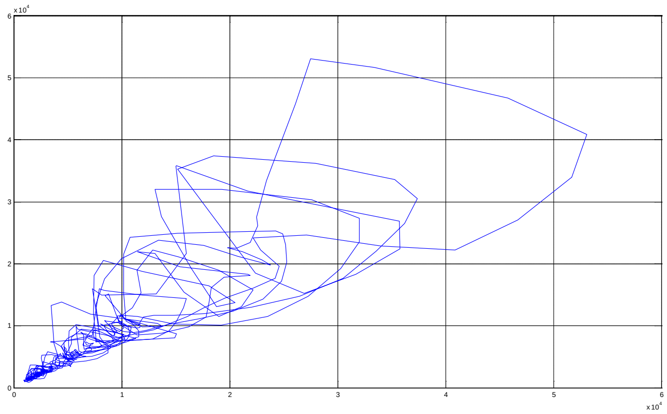

Figure 3. Attractor phase diagram of runoff series in 2D reconstructed phase space

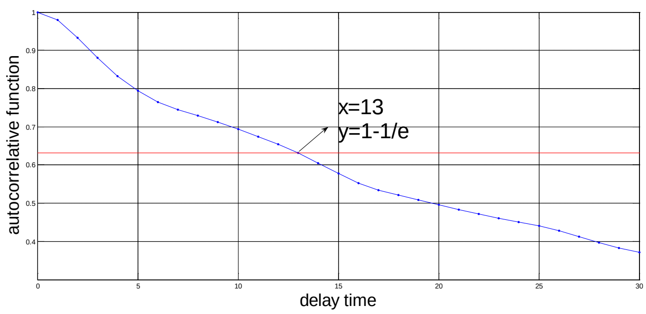

Figure 4. Delay time determination by mutual information method

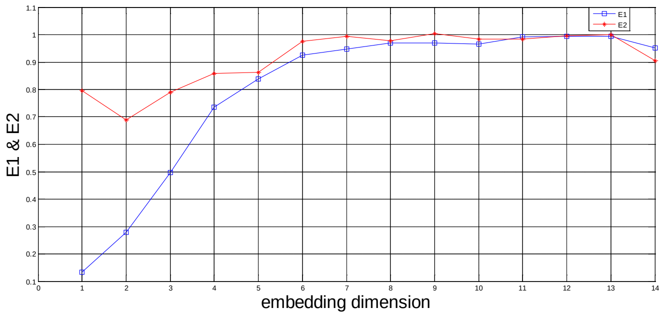

Figure 5. Embedding dimension of reconstructed phase space

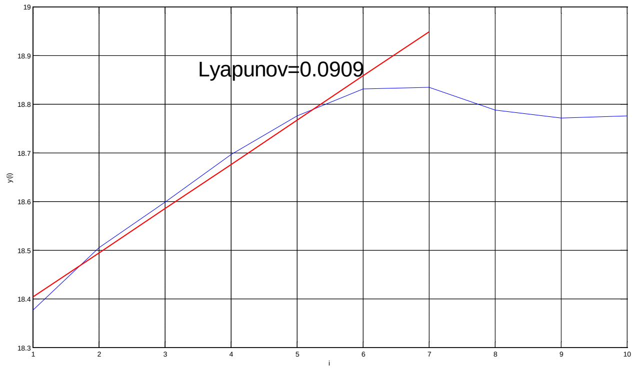

Figure 6. Lyapunov index of runoff series

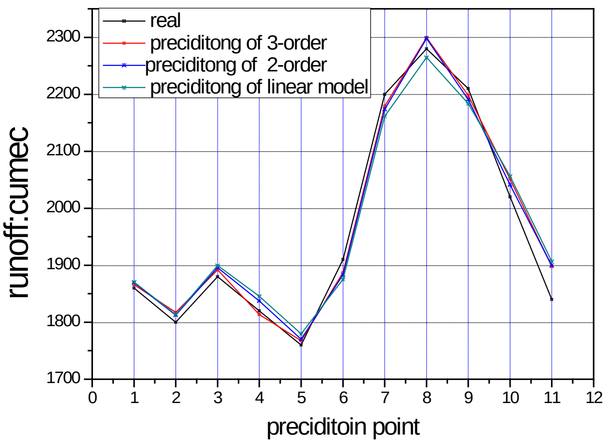

Figure 7. Prediction effect of models and comparison

Table 1. Relative error intervals of prediction model

prediction point	Relative error(%)		
	2-order Chebyshev	3-order Chebyshev	Linear
1	0.496	0.329	0.574
2	0.68	0.957	0.76
3	0.84	0.645	1.033
4	0.953	0.36	1.405
5	0.588	0.417	1.118
6	1.396	1.192	1.803

7	1.212	0.944	1.747
8	0.784	0.848	0.673
9	0.865	0.537	1.191
10	1.034	1.546	1.815
11	3.236	3.127	3.624
Mean	1.099	0.991	1.431

Table 2. Mean square error of prediction models

Model	2-order Chebyshev	3-order Chebyshev	Linear
MSE	0.142	0.136	0.18

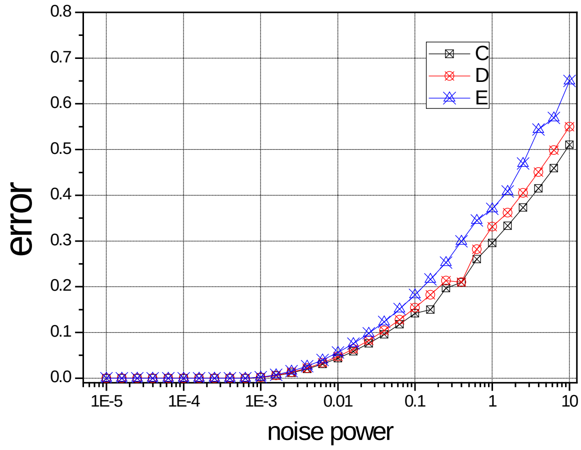

Figure 8. Relationship between noise power and model coefficient error

5. Conclusions

Due to influences from astronomy, climate and human activities, hydrologic system has evolved into a nonlinear complex system. As a result, traditional runoff prediction models based on probability mathematical statistics, random process and theory of time series couldn't depict runoff evolution trend objectively. This paper analyzes dynamics of runoff based on the chaos theory. It confirms that actual runoff has chaotic dynamics through a qualitative analysis on power spectra of runoff and geometric structure of attractor as well as a quantitative analysis on Lyapunov. This reflects that local prediction model based on chaos theory conforms to evolution law of runoff better.

Next, runoff prediction problems are studied in this paper. Combining with phase-space reconstruction technology in the chaos theory, a chaos local prediction model of runoff based on Chebyshev polynomials is proposed. The proposed model combines good approximation effect of Chebyshev polynomials and strong nonlinear system description capability of local prediction model, thus enabling to disclose internal relations and evolution mechanism of nonlinear complex hydrologic runoff system accurately. It overcomes heavy computation load and poor noise resistance of traditional local prediction models under high order of polynomials and big data size. This model is proved successful by a numerical simulation with actual runoff. Furthermore, this model has simple principle, clear idea, quick computation and high application values.

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